EUCLID

SIMPLIFIED

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COMPILED FROM THE MOST IMPORTANT FRENCH WORKS, APPROVED BY THE UNIVERSITY OF PARIS, AND THE MINISTER OF PUBLIC INSTRUCTION.

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65 CORNHILL, AND 12 PATERNOSTER ROW, LONDON.

1875.

PREFATORY REMARKS.

The work here offered for the use of schools goes over almost exactly the same ground as our Euclids. But in France, as in Germany, the text of Euclid has been long since discarded as cumbrous and wordy, and his theorems and problems have been reclassed and produced in less antiquated language than with us.

It is conceived that this facilitates the study of Geometry for boys. A few improvements are also admitted, taken from what is called Modern Geometry. This appears reasonable, as it is not natural that Geometry, like other sciences, should not have advanced in the space of two thousand years.

The present work gives what is best in French School Geometries, with certain additions from Modern Geometry.

It is anticipated that it will prove more practically useful than most other school books on the subject.

The theorems and problems are throughout taken from Amiot's "Éléments de Géométrie," Amiot's "Leçons Nouvelles de Géométrie," Legendre's "Éléments de Géométrie" (New Edition, 1872, by Blanchard, of the Polytechnic School), Rouché et de Comberousse's "Traité de Géométrie," Cortazar's "Geometria Elemental," and Professor Bos' (of the Lycée St. Louis) "Géométrie Elementaire."

April, 1875.

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EUCLID SIMPLIFIED.

BOOK I.

DEFINITIONS.

1. **Every** body occupies in indefinite space a determinate portion, called volume.

2. The *surface* of a body is the limit which separates it from surrounding space.

3. The place where two surfaces meet is called a line.

4. A point is the place where two lines intersect.

5. Volumes, surfaces, and lines are conceived independently of the bodies to which they belong.

6. The name of figure is given to volumes, surfaces, and lines.

7. Geometry has for its object the measure of the extension of figures, and the study of their properties.

8. A straight line is an indefinite line, which is the shortest between any two of its points.

It may be regarded as self-evident, that from one point to another only one straight line can be drawn, and that if two portions of a straight line coincide, these lines coincide throughout their whole extent.

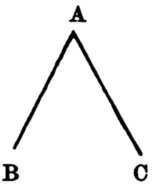
9. A broken line is a line composed of straight lines.

10. A curved line is one no portion of which is straight.

11. A plane is a surface, such that if any two points are taken in it, the straight line joining them is entirely in the plane.

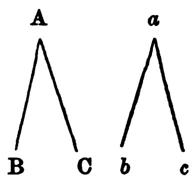
12. Every surface which is neither plane nor composed of plane surfaces is a curved surface.

13. The figure formed by two lines that intersect, as A B, A C, is called an angle. Point A is the summit or vertex of the angle; and lines A B, A C are its sides.

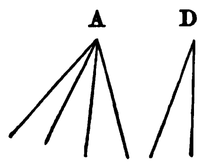


An angle is sometimes designated by the letter at the summit A. At other times it is designated by three letters, B A C, or C A B, placing the letter of the vertex in the middle.

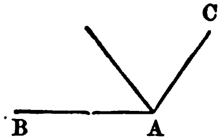
Two angles A and *a* are said to be equal when they can be made to coincide. Thus, supposing we apply angle *a* on A, so that *a b* should be applied to A B; if *a c* takes the direction A C, the sides of the two angles will coincide, and the two angles will be called equal.



An angle A is double, triple, &c., of the angle D if it includes between its sides two or three angles equal to angle D.

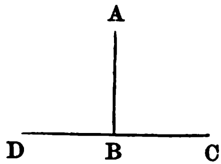


Angles, like other magnitudes, are therefore comparable to each other.

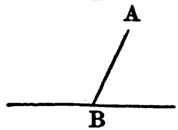


14. The size of an angle, for example B A C, depends only on the departure of its sides, which must always be conceived to be indefinitely prolonged. To have an idea of its dimensions, side A C is conceived to be at first applied to A B; then it is caused to turn round the summit A, till it has resumed its original position. The extent to which the straight line A C has turned is precisely what constitutes the size of the angle B A C.

15. Two figures are equal when they can be made to coincide, by applying them one to the other. In superposing two plane figures, this principle must be admitted —which will be demonstrated further on—that "*if three points in a plane are applied to another plane, these two surfaces coincide throughout their extent.*" Two figures are said to be equivalent when they have the same extent, without having the same form.



16. Two angles are said to be *adjacent* when they have the same summit B, a common side B A, and are placed on opposite sides of this line B A.



17. A straight line A B is *perpendicular* or *oblique* to another straight line, when it makes with the latter two adjacent angles, equal or unequal. In both cases the intersection B of the two straight lines is named the foot of the perpendicular or of the oblique line. Every angle of which one side is perpendicular to the other is named a right angle. An acute angle is smaller, an obtuse angle larger, than a right angle.

18. A theorem is the proposition of a truth which is not evident and has to be proved.

The enunciation of a theorem includes two parts, namely, an *hypothesis*, made in a certain case, and a *conclusion*, which is the consequence of the hypothesis. The reasoning that is made to deduce the conclusion from the hypothesis, when their dependence is not evident, is named the *demonstration* of the theorem.

19. Two theorems are *reciprocal* when the hypothesis and the conclusion of one are the conclusion and the hypothesis of the other. Thus the theorem — *If two angles are right angles, they are equal*—has for its reciprocal—*If two angles are equal, they are right angles.*

When the conclusion of a theorem agrees with more cases than that of the hypothesis, the reciprocal theorem may be false. We have an example of this in the theorem previously enunciated, for two angles may be equal without being right angles.

All propositions are *direct*, *reciprocal*, or *contrary*—all so closely connected that either of the two latter is a consequence of the other two. It is a direct proposition to prove, for instance, that all points in a certain figure, like the circumference of a circle, enjoy a certain property (*e.g.* the same distance from the centre). The reciprocal proposition shows that all points enjoying this property belong to the circumference. The contrary proposition shows that all points taken outside or inside the figure (the circumference) do not enjoy this property.

The direct and the reciprocal proofs are generally the simpler, and do not require a fresh construction.

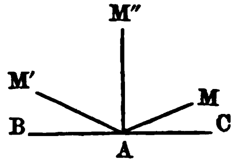
20. The term corollary of a theorem is given to any consequence drawn from a theorem.

21. Algebraic signs are used to abbreviate geometrical reasonings. Thus $A < B$ means that A is less than B —$A > B$ means that A is greater than B. The sign for addition is $+$, for substraction —, for multiplication $\times$. To describe half an angle it is usual to write $\frac{1}{2}A$. The square on a line is described: $\overline{AB}^2$ the cube, $\overline{AB}^3$ or $\overline{BC}^3$

The sign $\surd$ expresses the square root. Thus $\sqrt{A \times B}$ is the square root of the product A and B.

THEOREM I.

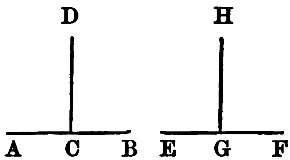
From a point in a straight line a perpendicular can be raised on this line, but only one.



Let it be supposed that a straight line A M, first placed upon the line B C, turns round the point A. It will then form two adjacent angles, M A C, M A B, of which one, M A C, is in the first instance very small, but will increase continually, while the other M A B, at first larger than M A C, will continually decrease till B.

The angle M A C, at first less than M A B, will therefore become at last greater than the latter angle; therefore there will be a position A M″ of the movable straight line A M, where these two angles M A C, M A B will be equal, and it is evident that there is only one such position.

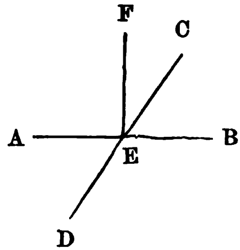
Corollary.—All right angles are equal. Let D C be a perpendicular to A B, and H G to E F. Then the angle



D C B is equal to the angle H G F. For if the straight line E F be laid upon the line A B, so that the point G falls upon C, G H will take the direction C D; otherwise, from the same point in a straight line two perpendiculars could be raised to the same straight line.

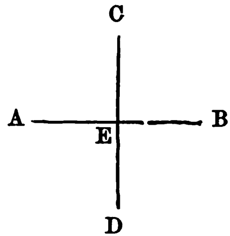
THEOREM II.

When a straight line A B *meets another* C D, *the sum of the two adjacent angles* A E C, B E C, *formed by these lines, is equal to two right angles.*

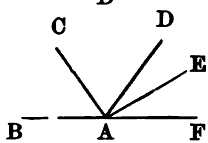


If the line C D is perpendicular to A B, the theorem is evident, since the adjacent angles A E C, B E C are then right angles.*

But if C D be not perpendicular, from the point E, let the perpendicular E F be raised on A B. Then the obtuse angle† A E C is larger than the right angle A E F, by their difference C E F, while the acute † angle B E C is less than the right angle B E F by the same difference, angle C E F. Therefore the sum of the two adjacent angles A E C, B E C is equal to the sum of the two right anglesA E F, B E F.



Corollary I.—If one of the angles A E C, B E C is a right angle, the other will be so likewise.



Corollary II.—All the consecutive angles B A C, C A D, D A E, E A F on one side of the line B F are equal to two right angles. For their sum is equal to that of the two adjacent angles B A C, C A F.

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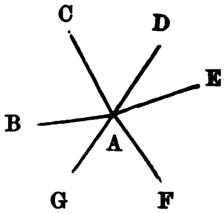
* See diagram to Corollary I. † Def. 17.



Corollary III. — The sum of any number of angles B A C, C A D, D A E, E A F, F A G, B A G formed round a point A by the straight lines A B, A C, A D, A E, A F, A G drawn from that point, is equal to four right angles.

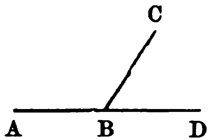
For if at point A we form four right angles by means of two lines perpendicular to each other, their sum will be evidently equal to that of the above angles, B A C, C A D, &c.

DEFINITION.

22. Two angles are complementary if they are together equal to one right angle. They are supplementary if they are together equal to two right angles.

THEOREM III.

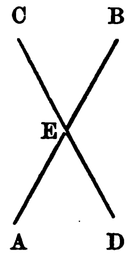
If two adjacent angles A B C, C B D *are supplementary, their sides that are not common, as* A B, B D, *are in one straight line.*



For the prolongation of the straight line A B, beyond the point B, ought to make with B C an angle equal to the supplement of the angle A B C, *i.e.* equal to the angle C B D. Therefore it coincides with the straight line B D.

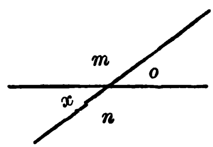
THEOREM IV.

If two straight lines A B, C D *intersect, the angles* A E C, B E D *opposed at the summit are equal.*



The adjacent angles AEC, C E B made by the straight lines A B, E C are equal to two right angles (Theorem II.); also the angles C E B, D E B are together equal to two right angles. Therefore A E C, C E B taken together are equal to C E B, D E B taken together. If from these equals we take the common angle C E B, the remaining angles A E C, D E B will be equal. In like manner it can be proved that A E D, C E B are equal.

The demonstration may be given in a shorter form, thus:



Let the opposite angles be designated by the letters o and x, m and n. These are taken together:—

$$m + o = 2 \text{ right angles.}$$
$$m + x = 2 \text{ right angles.}$$
Then $m + o = m + x$.
But $m = m$.
Therefore $o = x$.

DEFINITIONS.

23. A triangle is the portion of a plane inclosed by three lines that cut each other, two and two, and called its sides.

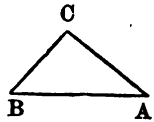
24. A triangle is isosceles when it has two sides equal. (In Greek ἴσος, equal, σκέλος, leg.) The base of an isosceles triangle is the unequal side.

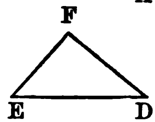
25. A triangle is equilateral when its three sides are equal.

26. A triangle is right-angled when it has a right angle, and the side opposite to the right angle is named the hypothenuse.

THEOREM V.

Two triangles are equal when they have an equal angle included between two equal sides, each to each.





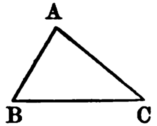
Let angle A be equal to angle D, side A B equal to D E, side A C equal to D F. Then the triangles A B C, D E F will be equal.

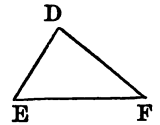
For these triangles can be placed one on the other, so as perfectly to coincide. And first, if side D E be placed on its equal A B, point D will fall on A, and point E on B. But since the angle D is equal to angle A, as soon as side D E is placed on A B, side D F will take the direction A C. Moreover, D F is equal to A C; therefore point F will fall on C, and the third side E F will cover exactly the third side B C; therefore triangle D E F is equal to triangle A B C.

Corollary.—From the fact that three parts are equal in two triangles, namely, angle A = D, side A B = D E, and side A C = D F, it may be concluded that the other three parts are also equal, namely, angle B = E, angle C = F, and side B C = E F.

THEOREM VI.

Two triangles are equal when they have an equal side adjacent to two angles, equal each to each.





Let A B C, D E F be two triangles, having the side B C equal to E F, the angle A B C equal to D E F, and the angle A C B equal to D F E. These triangles will then be equal.

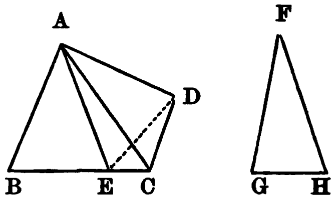
For applying triangle D E F to triangle A B C, the side E F can be made to coincide with the side B C, equal to it, by placing point E on point B, and point F on point C. In that case, the side E D takes the direction B A, because angles D E F, A B C are equal; side F D takes likewise the direction C A, because of the equality of angles D F E, A C B. Consequently point D, common to the two straight lines E D, F D, is placed at the intersection A of those two straight lines B A, C A, and the triangles A B C, D E F coincide throughout their extent.

Corollary.—When two triangles have an equal side adjacent to two angles, equal each to each, the sides opposite to the equal angles are also equal. The same is the case with the angles opposite to the equal sides.

THEOREM VII.

If two sides of a triangle are equal to two sides of another, each to each, and if at the same time the angle included by the first is larger than the angle included by the second, then the third side of the first triangle will be greater than the third side of the second.

In the triangles A B C, F G H, let the angle B A C be greater than G F H, but the sides A B, A C equal



to F G, F H, each to each. Then side B C will be greater than G H.

Make the angle C A D equal to G F H, take A D = G F and join C D. The two triangles A C D, G F H are equal as having an equal angle included between equal sides.* It is therefore sufficient to prove that B C is greater than C D.

Divide the angle B A D into two equal parts by the line A E. This line will fall inside the greater angle B A C; then draw the line D E; the two triangles B A E, E A D will be equal as having an equal angle included between two equal sides.* Therefore $BE = ED$. But in the triangle E D C, $CD < ED + EC$. Substituting B E for E D, this gives $CD < BE + EC$, or $CD < BC$.

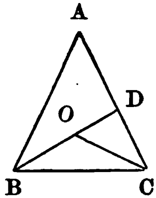
Reciprocally, if the sides A B, A C of the triangle A B C are equal to the two sides F G, F H of the triangle G F H or A C D; if, moreover, the third side C B of the first triangle is greater than the third side G H of the second, the angle B A C will be greater than the angle G F H.

For if angle B A C were less than C A D, it has been seen that C B would be less than C D, which is against the hypothesis; and if the angle B A C were equal to C A D, this would give $CB = CD$, also against the hypothesis.

* Theorem V.

THEOREM VIII.

If from a point O taken inside the triangle A B C, the straight lines O B, O C be produced to the extremities of a side B C, the sum of these straight lines will be less than that of the other two sides A B, A C, enveloping them.

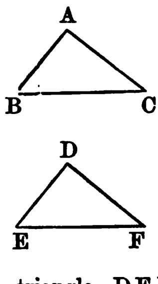


Let B O be prolonged till it meets A C at D; the straight line O C is shorter than O D + D C :* adding to both B O, this gives B O + O C < B O + O D + D C, or B O + O C < B D + D C.

Similarly B D < B A + A D; adding to both sides D C. this gives B D + D C < B A + A C. But it has been seen that B O + O C < B D + D C; therefore *à fortiori* B O + O C < B A + A C.

THEOREM IX.

Two triangles are equal when they have the three sides equal, each to each.



Let there be two triangles A B C, D E F, having the side A B equal to D E, the side A C equal to D F, and the side B C equal to E F. In that case two angles such as A, D, opposite to the equal sides B C, E F, are equal.

For the two sides A B, A C of the triangle A B C being equal respectively to the sides D E, D F of the triangle D E F the third sides B C, E F, of those triangles can only be equal if the angles A and D opposite to those sides are themselves

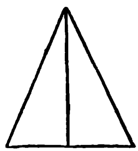
* Def. 8.

equal.* Now, B C is equal to E F by hypothesis; therefore the angle A is also equal to D. Then the triangles A B C, D E F having an equal angle included between two equal sides, each to each, are equal.†

Corollary.—If two triangles have the three sides equal, each to each, the angles opposite to the equal sides are equal.

THEOREM X.

In an isosceles triangle, the angles opposite to the equal sides are equal.



Let side A B = A C; then angle C = B. Draw the line A D from the *vertex* A to the point D at the centre of the *base* B C; the two triangles A B D, A C D will have their three sides equal, each to each, namely A D common, A B = A C by hypothesis, and BD = DC by construction; therefore, by the preceding theorem, angle B is equal to angle C.

Corollary I.—An equilateral triangle is at the same time equiangular, that is, it has its angles equal.‡

Corollary II.—The equality of the triangles A B D, A C D proves at the same time that the angle B A D = D A C, and that the angle B D A = A D C; therefore these two latter are right angles; § *therefore the line drawn from the summit of an isosceles triangle to the centre of the base is perpendicular to that base, and divides the angle at the summit into two equal parts.*

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* Theorem VII. † Theorem V.

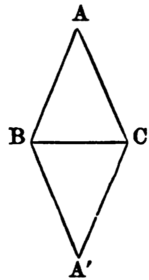
‡ This direct proof of the equality of the angles at the base of an isosceles triangle is shorter and simpler than that of Euclid (5, 1st book). French geometries do not treat of the angles under the base.

§ Theorem II., Cor. 1. Theorem I. and Def. 17.

THEOREM XI.

*If a triangle has two equal angles, the sides opposite to these angles are also equal, and the triangle is isosceles.**

Let A B C be a triangle of which the angles A B C, A C B are equal; then side A C opposite to angle A B C is equal to side A B opposite to angle A C B.



On the side B C make the triangle B C A′ equal to the triangle A B C, making angle B C A′ equal to angle C B A, and side C A′ equal to side B A. These triangles are therefore equal, since they have an equal angle included between two equal sides, each to each.† Therefore angle C B A′ opposite to side C A′ is equal to angle B C A opposite to side B A, and consequently equal to angle C B A. This being established, fold the figure over the line B C, and

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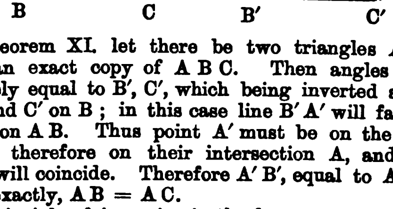
* Theorems X. and XI. are demonstrated by the inversion as well as superposition of angles.



In Theorem XI. let there be two triangles A B C and A′ B′ C′, an exact copy of A B C. Then angles B, C are respectively equal to B′, C′, which being inverted are placed B′ on C and C′ on B; in this case line B′ A′ will fall on A C, and C′ A′ on A B. Thus point A′ must be on the two lines A C, A B, therefore on their intersection A, and the two triangles will coincide. Therefore A′ B′, equal to A B, covering A C exactly, A B = A C.

This principle of inversion is the key to all the properties of the isosceles triangle.

† Theorem V.

place the triangle A′BC on triangle ABC. Then the side BA′ takes the direction BA, because of the equality of the angles CBA′, CBA, and the side CA′ falls on CA, since angles BCA′, BCA are by hypothesis also equal. Consequently the summit A′ will coincide with summit A, and side CA be equal to side CA′ or BA.

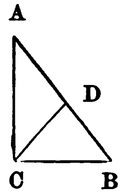
COROLLARY.—*An equiangular triangle is equilateral.* For the sides of this triangle are equal, as they are opposite to equal angles.

THEOREM XII.

If a triangle has two unequal angles, the side opposite to the greater angle is greater than the side opposite to the other angle.

1. In triangle CAB, let angle C > B; then the side AB opposite to angle C, is greater than side AC opposite to angle B.

Make angle BCD = angle B; then in the triangle BDC, BD will be = DC.* But the straight line AC is shorter than AD + DC, and AD + DC = AD + DB = AB; therefore AB is greater than AC.



2. Let side AB > AC; then angle C opposite to side AB will be greater than angle B opposite to side AC.

For if C < B, it would follow, by (1), that AB < AC, which is contrary to the hypothesis. If C = B, then AB = AC, also against the hypothesis; therefore angle C must be greater than B.

THEOREM XIII.

From a given point outside a straight line, only one perpendicular can be drawn to the same straight line.

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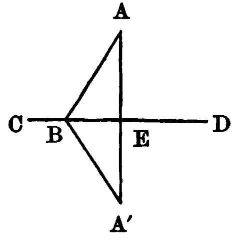
* Theorem XI.



Let us suppose that from point A two perpendiculars A E, A B can be drawn to C D.

Let one of these perpendiculars A E be produced by a length E A′ = A E, and join A′ B.

Triangle A E B is equal to triangle A′ E B, for the angles A E B, A′ E B are right angles; side A E = A′ E, and side B E is common.* Therefore ABE=EBA′; but A B E is a right angle (hypothesis), therefore E B A′ is also a right angle.

But if the adjacent angles A B E, E B A′ are together equal to two right angles, the line A B A′ must be a straight line.† Therefore from point A to point A′ two straight lines could be drawn, which is impossible.‡ Therefore, &c.

THEOREM XIV.

If from a point A *situated outside a straight line* D E *a perpendicular* A B *be drawn to that straight line* D E, *and also different oblique lines* A E, A C, A D, *&c., to different points of the same line* D E ; *then—*

1. *The perpendicular* A B *will be shorter than any oblique.*

2. *The two obliques* A C, A E *on each side of the perpendicular at equal distances* B C, B E *will be equal.*

3. *Of two obliques* A C *and* A D, *or* A E *and* A D, *drawn at different distances from* A B, *that which deviates the most from the perpendicular will be the longest.*

Produce the perpendicular A B by a length B F = A B, and join F C, F D.

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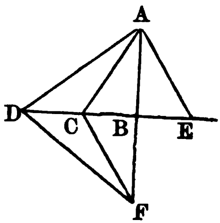
* Theorem V. † Theorem III. ‡ Def. 8.



1. Triangle B C F is equal to triangle B C A, for the right angle C B F = C B A, side C B is common, and side B F = B A; therefore C A = C F. But A B, half of A B F, is shorter than A C, half of A C F; * therefore, first, the perpendicular is shorter than every oblique.

2. If B E be by hypothesis equal to B C, A B is moreover common, and angle A B E = A B C; therefore triangle A B E is equal to triangle A B C; therefore the sides A E, A C are equal; therefore, secondly, two obliques which deviate equally from the perpendicular are equal.

3. In triangle D F A, the sum of the lines A C, C F is less than the sum of the sides A D, D F; * therefore A C, half of the line A C F, is shorter than A D, half of A D F; † therefore, thirdly, the obliques that deviate the most from the perpendicular are the longest.

Corollary I.—The perpendicular measures the true distance from a point to a line, since it is shorter than every oblique.

Corollary II.—From one and the same point three equal straight lines cannot be drawn to another straight line; for if that were the case, there would be on the same side of a perpendicular two equal obliques, which is impossible.

THEOREM XV.

If through the point C, *the centre of the straight line* A B, *the perpendicular* E F *be raised on that straight line, first, each point of the perpendicular will be equally distant from the two extremities of the line* A B; *secondly, every point situated outside the perpendicular will be unequally distant from the same extremities* A *and* B.

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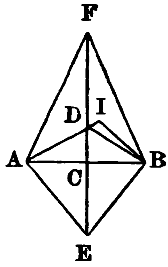
* Theorem XIII. † Theorem VIII.



1. For, first, since by hypothesis A C = C B, the two obliques A D, D B deviate equally from the perpendicular;* therefore they are equal. The same is the case with the two obliques A E, E B, of the other two A F, F B, &c.; therefore, first, every point of the perpendicular is distant equally from the extremities A and B.

2. Let I be a point outside the perpendicular; if lines I A, I B be drawn, one of these lines will cut the perpendicular at D, from which if we draw D B, this gives D B = D A. But the straight line I B is less than the broken line I D + D B, and I D + D B = I D + D A = I A; therefore I B < I A; therefore, secondly, every point outside the perpendicular is unequally distant from the extremities A and B.

Remark.—In plane geometry, the term *geometrical locus* † is given to a line of which all the points enjoy a common property, to the exclusion of all the other points of the plane.

The line E F is therefore the *geometrical locus* of the points equally distant from the points A and B.

THEOREM XVI.

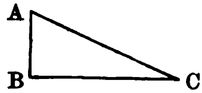
Two right-angled triangles are equal if they have the hypothenuse equal and another side equal.

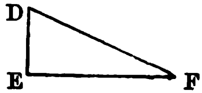
---

* Theorem XIV.

† Todhunter (School Euclid, Appendix, p. 328) describes a *locus* as consisting of all the points which satisfy certain conditions, and of these points alone.

The term *locus* has been retained from our usual terminology, though the French term *lieu*, translated by *place*, seems to be preferable, as more modern and English.

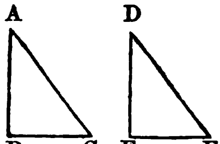




Let A B C, D E F be two right-angled triangles, and B, E their right angles; I suppose the hypothenuse A C equal to D F, the side A B equal to D E, and it is inferred that these two triangles are equal. To prove it, let triangle D E F be placed on triangle A B C, making the equal sides D E, A B coincide, placing point E on point B, point D on point A. Then side E F takes the direction B C, because of the equality of the right angles E and B, and the hypothenuse D F is applied to A C, because these two equal lines are obliques to B C, and situated on the same side of the perpendicular A B.* The triangles D E F, A B C are therefore equal, since their sides coincide.

THEOREM XVII.

Two right-angled triangles are equal when they have the hypothenuse and an angle equal.



Let A C = D F and angle A = D. Placing D E F on A B C, so that D F covers A C, angle D being equal to angle A, D E will take the direction A B, and at the same time F E will take the direction C B, for otherwise it would be possible from point C to let drop two perpendiculars on A B. Therefore point E will fall on B, and the two triangles will perfectly coincide.

* Theorem XIV., Cor. II.

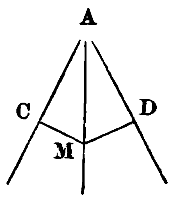
THEOREM XVIII.

1. *Every point* M *taken on the bisector* * *of an angle* C A D *is equally distant from the sides of the angle.*

2. *Every point* M *taken in the angle* C A D, *and equally distant from the sides* A C, A D, *belongs to the bisector of that angle.*

1. From point M let drop the straight lines M D and M C respectively perpendicular to A D and A C. The right-angled triangles M A D, M A C are equal, for they have the hypothenuse M A common, and the angles M A D, M A C equal by hypothesis; therefore M D = M C.†

2. *Reciprocally*, if the perpendiculars M D, M C are equal, the right-angled triangles M A D, M A C will also be equal, as having the hypothenuse M A common, and the sides M D, M C equal by hypothesis; therefore the angle M A D = M A C. ‡



It results from this that every point taken in the angle C A D outside the bisector A M is unequally distant from the two sides.

Scholium.—The bisector of an angle is the geometrical locus of the points situated in the interior of this angle, and which are equally distant from its sides.

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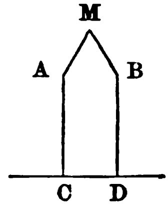
* The *bisector* of an angle is the line dividing the angle into two equal parts.

† Theorem XVII. ‡ Theorem XVI.

THEORY OF PARALLELS.

THEOREM XIX.

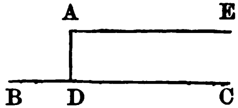
Two straight lines A C, B D, *perpendicular to the same straight line* C D, *are parallel.*



For if they met at a point M, for example, it would be possible from that point to drop two perpendiculars on C D.*

THEOREM XX.

Through a point A *situated outside a straight line* B C, *a parallel, and only one, can be drawn to that line.*



From point A draw the perpendicular A D to the straight line B C, and the perpendicular A E to the straight line A D. The lines A E and B C are parallel, since both are perpendicular to A D; † hence through point A a parallel can be drawn to the straight line B C; and it may be admitted ‡ *that only one parallel can be drawn to it.*

Corollary.—*If two straight lines are parallel, every straight line which meets one of them will also, if produced, meet the other.*

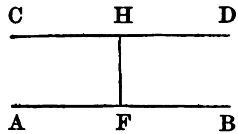
---

* Theorem XIII. † Theorem XIX.

‡ It is well known to geometricians that in all the reasonings on parallels a fundamental proposition has to be taken for granted as self-evident. The above is assumed by Legendre, p. 20, *Eléments de Géométrie* (1871); Amiot, p. 24, *Eléments de Géométrie;* Rouché and De Comberousse, p. 29, *Traité de Géométrie Élémentaire,* &c.

THEOREM XXI.

If two straight lines C D, A B *are parallel, every straight line* F H *perpendicular to one of them* A B *is perpendicular to the other* C D.

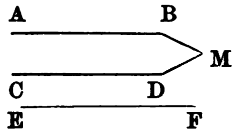


It is in the first place evident that F H must meet C D, otherwise it would be possible through the point F to draw two parallels to C D.

Lastly, C D is perpendicular to F H, for if the line C D were oblique to F H, at point H a perpendicular could be raised to F H, which would be parallel to A B, and thus there would be two straight lines, passing by the point H and parallel to A B. *

THEOREM XXII.

Two straight lines A B, C D *parallel to a third* E F *are parallel to each other.*

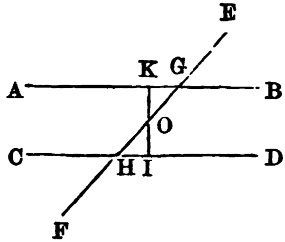


For if the two lines A B, C D met at a point M, it would be possible from that point to draw two parallels to E F.*

THEOREM XXIII.

When two parallel straight lines A B, C D *are met by a secant* E F, *the four acute angles resulting from it are equal to one another, and also the four obtuse angles.*

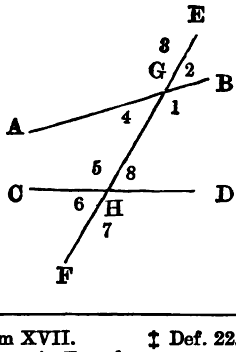
* Theorem XX.



Let G and H be the points where the straight line E F meets the parallels A B, C D. The acute angles A G H, E G B are equal because they are opposed at the vertex.* The same is the case with the acute angles G H D, C H F. To prove the equality of the four acute angles formed by the straight lines A B, C D, E F, it is sufficient therefore to prove that the angle A G H is equal to G H D.

From the centre O of the line G H let drop the perpendicular I K on the two parallels A B, C D. The right-angled triangles G K O, H I O are equal,† because they have the hypothenuses O G, O H equal, and the acute angles G O K, I O H equal as opposed at the summit; therefore the angle O G K is equal to the angle O H I. It results from the equality of the four acute angles that the four obtuse angles are equal to one another, for each obtuse angle has for supplement ‡ one of the four acute angles.

DEFINITIONS.



27. When two straight lines A B, C D are cut by a *transversal* § E F, there are eight angles formed at the points of intersection G and H.

The four angles (1), (4), (5), (8) included between the two straight lines A B and C D are called internal angles. The four others are called external angles.

Two angles such as (1)

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* Theorem IV. † Theorem XVII. ‡ Def. 22.

§ The term transversal, common in French geometry and

and (5), situated on opposite sides of the secant, or transversal, internal and not adjacent, are called alternate internal.

Two angles such as (8) and (2), situated on the same side of the secant, one internal and the other external and not adjacent, are called corresponding angles.

Lastly, angles such as (2) and (6), situated on opposite sides of the secant, external and not adjacent, are called alternate external.

From this we obtain a new enunciation of Theorem XXIII.

THEOREM XXIV.

If two straight lines are parallel, they will form with any secant—

1. *Equal alternate internal angles.*
2. *Equal alternate external angles.*
3. *Equal corresponding angles.*
4. *Supplementary internal angles on the same side.*
5. *Supplementary external angles on the same side.*

---

new to English schools, explains itself: *trans*, across, *verto*, I turn. It is a secant or straight line cutting two or more other lines, like the sides of a triangle.

The application of transversals is largely adopted in French elementary geometries, and is useful in the analysis of proportions and ratios. Menelaus, a Greek geometer (100 years before Ptolemy), reasoned on the principle of transversals. John de Ceva, an Italian geometer of the 16th century, applied the same principle, and Carnot gave a *Théorie des Transversales* in his *Géométrie de Position*. Amiot says: "When you draw a straight line on the plane of a triangle, it can meet the three sides or be parallel to one of them. In these two cases it is given the name of *transversal.*"—*Leçons Nouvelles*, p. 94. (*See* Appendix.)

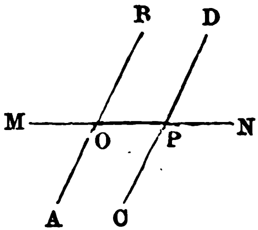
Todhunter's definition of transversals is: Any line, straight or curved,. which cuts a system of other lines is called a *transversal.*—Appendix, "On Modern Geometry," p. 335, *School Euclid*, 1867.

For two alternate internal or alternate external or corresponding angles are either both acute or obtuse, and therefore equal to one another.* But of two internal or external angles on the same side, one is acute and the other obtuse; these angles are therefore supplementary.† (See diagram, Theorem XXV.)

THEOREM XXV.

Reciprocally, two straight lines A B, C D *are parallel when they make with the secant* M N—

1. *Equal alternate internal angles.*
2. *Equal alternate external angles.*
3. *Equal corresponding angles.*
4. *Supplementary internal angles on the same side.*
5. *Supplementary external angles on the same side.*



Let it be granted that the alternate internal angles A O P, D P O are equal. Then the straight lines A B, C D are parallel.

For the straight line A B and its parallel, drawn through the point P where the secant M N meets C D, make with M N equal alternate internal angles.‡ Now, one of these angles is A O P, therefore the other is the angle D P O, since they are by hypothesis the position of two alternate internal angles, and since they are equal. Therefore the parallel drawn from point P to the straight line A B is no other than the line C D.

It could be proved by an analogous reasoning in the four other cases that A B is parallel to C D.

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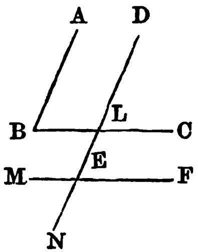
* Theorem XXIII. † Def. 22. ‡ Theorem XXIV.

Corollary.—The *contrary* * propositions to the two preceding ones are true. They include this special theorem:—Two *straight lines will meet* when *they make with a secant two internal angles on the same side, of which the sum is less than two right angles.* In his celebrated Treatise on Geometry, Euclid † demands the admission of this theorem as evident, and he makes it the basis of his theory of parallel lines; accordingly, this theorem is known as the postulate of Euclid. In the present treatise we have substituted the following:—"*Through a given point only one parallel can be drawn to a given straight line*" (Theorem XX.).

THEOREM XXVI.

Two angles which have their sides parallel are equal or supplementary.



1. Let A B C, D E F be two angles, of which the sides are parallel and turned in the same direction. These angles will be equal. For the angles D L C, D E F are equal as corresponding angles.‡ But for the same reason D L C = A B C, therefore A B C = D E F.

2. Let there be two angles A B C, M E N, of which the sides are parallel, but turned in an opposite direction. These angles will be equal; for M E N = D E F, and D E F = A B C.

---

* If in the ennunciation of a proposition a negation of an hypothesis be added to the consequence, the *contrary* proposition is found. For example, to the following proposition, "Two right angles are equal," corresponds the contrary proposition: "If two angles be not right angles, they are not equal"—evidently a false conclusion. — See Definition 19. Book I.

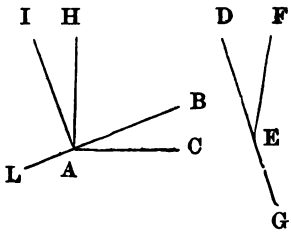
† A Greek geometer, who lived about B.C. 320.

‡ Theorem XXIV.

3. Lastly, two angles A B C, D E M, of which the sides are parallel, but of which two sides, B A and E D, are turned the same way, and the two others, B C and E M, in a contrary direction, are supplementary; for D E M is the supplement of D E F, and D E F = A B C.

THEOREM XXVII.

If two angles have their sides perpendicular, each to each, these angles will be equal or supplementary.

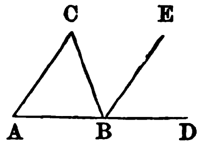


Let B A C, D E F be two angles, of which the sides are perpendicular, each to each. Draw through the point A a line A I perpendicular to A B, and a straight line A H perpendicular to A C; the straight lines A I, A H will be respectively parallel to the straight lines D E, E F, and turned in the same direction; therefore the angle I A H is equal to D E F; but I A H + H A B = 1 right angle, and B A C + H A B = 1 right angle. Therefore I A H = B A C.

Scholium.—If the angle formed by the straight line E F and the prolongation of D E were examined, it would be found that the angle F E G is supplementary to the angle B A C.

THEOREM XXVIII.

The sum of the angles of any triangle is equal to two right angles.



In the triangle A B C, let the side A B be produced, and through the summit B let the straight line B E be drawn parallel to the opposite side A C.

The angles A C B, C B E are equal as alternate internal in relation to the parallels A C, B E and the secant B C * (XXIII.); the angles C A B, E B D are also equal as corresponding angles in relation to the same parallels and to the secant A B. Therefore the sum of the three angles A B C, A C B, C A B of the triangle is equal to the sum of the adjacent angles A B C, C B E, E B D formed on the same straight line A D; that is, it is equal to two right angles.†

Corollary I.—The angle C B D made by the side B C with the prolongation B D of the side A B is *exterior* to the triangle. Hence results this theorem :—*An angle* C B D *exterior to a triangle* A B C *is equal to the sum of the interior angles* C A B, A C B *which are not adjacent to it.*‡

Corollary II.—A triangle can only have one right or obtuse angle ; then the two other angles are acute. The acute angles of a right-angled triangle are complementary.

Corollary III.—Each angle of an equilateral triangle is equal to two-thirds of a right angle.

Corollary IV.—If two angles of a triangle are respectively equal to two angles of another triangle, the third angle of the first triangle is also equal to the third angle of the second.

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* Theorem XXIV.

† Theorem II., Book I.

‡ This corollary to Theorem XXVIII. is made a part of Proposition 32 in Euclid (Book I.).

DEFINITIONS.

28. The term *polygon* is applied to a portion of a plane terminated by straight lines. These lines are named the sides of the polygon, and their sum total makes up the outline or perimeter of the figure.

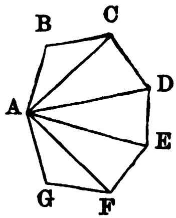
A polygon which has only three sides is a *triangle.* A polygon of four sides is named a *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon,* &c.

A polygon is *convex* when it is all on the same side of each of the straight lines that bound it prolonged indefinitely. In the opposite case it is styled *concave.*

The term diagonal of a polygon is given to the straight line which joins two non-consecutive summits of that polygon.

THEOREM XXIX.

*The sum of the interior angles of a polygon is equal to twice as many right angles as the figure has sides, minus two.**



Through one of the summits A of the polygon A B C D E F G, let diagonals be drawn to all the non-adjacent summits. The polygon will be divided into as many triangles as it has sides, minus two; for these different triangles can be considered as having for their common summit the point A, and for bases the different sides of the polygon, except the two last triangles, which each of them contain two sides of the polygon. It may be also seen that the sum of the angles of these triangles is equal to the sum of the angles

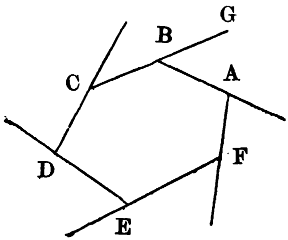
* This proposition is made a corollary of Proposition 32, Book I., Euclid.

of the polygon; therefore this last sum is equal to as many times two right angles as there are sides minus two. If n is used to represent the number of the sides of the polgyon, the sum of the angles will be: $2 \times (n - 2)$ or $2n - 4$.

THEOREM XXX.

The sum of the angles made outside a convex polygon by prolonging its sides in the same direction is equal to four right angles.

Each angle exterior to the polygon A B C D E F, such as the angle A B G, being the supplement of the



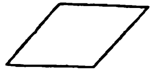
interior angle A B C adjacent to it (IV.), the sum of the exterior and interior angles is equal to as many times two right angles as the polygon has summits or sides. This sum is therefore equal to $2n$ right angles, n being the number of sides to the polygon. But the interior angles are equal together to $2n - 4$ right angles (XXIX.); therefore the sum of the external angles is equal to the excess of $2n$ right angles over $(2n - 4)$, that is, equal to four right angles.

Corollary.—A convex polygon has not more than three internal angles that are acute, for it cannot have more than three obtuse external angles.

DEFINITIONS.

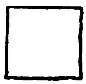
29. A *parallelogram* is a quadrilateral, of which the opposite sides are equal.

30. A *lozenge** is a quadrilateral which has all its sides equal. (It is also a parallelogram.)



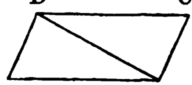
31. A *rectangle* is a parallelogram, of which all the angles are right angles.

32. A *square* is a rectangle, of which all the sides are equal, or a lozenge, of which all the angles are right angles.



THEOREM XXXI.

The opposite sides of a parallelogram are equal, also its opposite angles.



Draw the line BD, forming the diagonal of parallelogram ABCD. Then the two triangles ADB, DBC have the side BD common; moreover, because of the parallels AD, BC, angle ADB = DBC (XXIV.), and because of the parallels AB, CD, angle ABD = BDC; therefore the two triangles ADB, DBC are equal (VI.); therefore the side AB opposite to the angle ADB is equal to the side DC opposite to the equal angle DBC, and in like manner the third side AD is equal to the third BC; therefore the opposite sides of a parallelogram are equal.

---

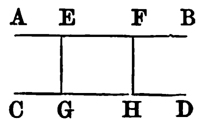
* Called rhombus in the old-fashioned Euclid.

In the second place, from the equality of the same triangles, it follows that angle A is equal to angle C, and also that angle A D C, made up of the two angles A D B, B D C, is equal to the angle A B C, made up of the two angles D B C, A B D; therefore the opposite angles of a parallelogram are equal.

Corollary I.—Therefore two parallels A B, C D included between two other parallels A D, B C are equal.

Corollary II.—Two parallels are everywhere equally distant.

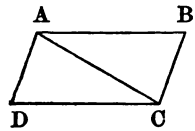
For C D and A B being parallels, let drop from points H and G, H F and G E, perpendiculars on A B. These straight lines will be parallel, and will be equal as being included between parallels.



THEOREM XXXII.

A quadrilateral, of which the opposite sides or angles are equal, is a parallelogram.

1. Let the quadrilateral A B C D have side A B equal to side D C, and side B C equal to A D. Then the opposite sides of this quadrilateral are parallel.



The diagonal A C divides the figure A B C D into two equal triangles, because they have the three sides equal, each to each (IX.); therefore the angle B A C opposite to the side B C is equal to the angle A C D opposite to the side A D. Now these two angles are alternate internal in relation to the two straight lines A B, C D, and to the secant A C; therefore A B is parallel to C D (XXV.); and it can be proved in like manner that B C is parallel to A D.

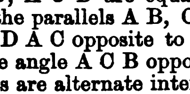
2. The quadrilateral A B C D, of which the opposite angles are equal, is also a parallelogram.

For, by the hypothesis, the sum of the two consecutive angles D A B, A B C is equal to the half of the sum of the four angles of the quadrilateral, that is, to two right angles (XXIX.). Now these angles are internal in relation to the two lines A D, B C, and situated on the same side of the secant A C; therefore the side A D is parallel to B C (XXVI.). In like manner the side A B is parallel to D C.

Corollary.—The lozenge* is a parallelogram, since its opposite sides are equal.

THEOREM XXXIII.

Every quadrilateral which has two opposite sides equal and parallel is a parallelogram.

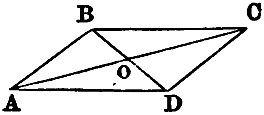


Let the quadrilateral A B C D have the side A B equal and parallel to D C. Draw the diagonal A C. This line divides the quadrilateral into two triangles, A B C, A D C, which are equal (Theorem V.); for the side A C is common, the side A B is equal to D C, by hypothesis, and the angles B A C, A C D are equal as alternate internal, in relation to the parallels A B, C D, and to the secant A C. The angle D A C opposite to the side D C is equal, therefore, to the angle A C B opposite to the side A B. But these angles are alternate internal in relation to the straight lines A D, B C, and to the secant A C. Consequently, A D is parallel to B C (Theorem XXV.), and the quadrilateral A B C D is a parallelogram.

* See Definition 30.

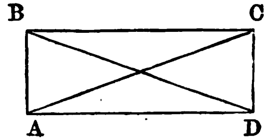
THEOREM XXXIV.

The two diagonals A C, D B *of a parallelogram cut each other mutually into two equal parts.*



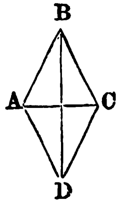
For, comparing triangle A D O to triangle C O B, it is seen that side A D = C B, the angle A D O = C B O (XXIII.), and the angle D A O = O C B; therefore these two triangles are equal; therefore A O, the side opposite to the angle A D O, is equal to O C, the side opposite to the angle O B C; therefore also D O = O B.

Corollary I.—*The diagonals of a rectangle* A B C D *are equal.*



For the triangles A D C, B A D, which have a right angle included between two sides, equal each to each, are equal (V.); and the diagonal A C, opposite to the right angle A D C is equal to the diagonal B D, opposite the right angle B A D.

Corollary II.—*The diagonals of a lozenge* A B C D *are perpendicular one to the other.*

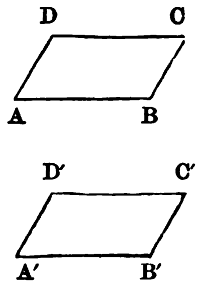


For the diagonal B D, of which the two points B, D are equally distant from the extremities of the diagonal A C, is perpendicular to that line (XV.), and divides it into two equal parts.

Corollary III.—*The diagonals of a square are equal and perpendicular to one another.* For the square is at once a rectangle and a lozenge.

THEOREM XXXV.

Two parallelograms are equal when they have an equal angle included between two equal sides, each to each.



Let A B C D, A′ B′ C′ D′ be two parallelograms, having the angle A equal to the angle A′, and the sides A B, A D respectively equal to the sides A′ B′, A′ D′. Then these quadrilaterals are equal.

For, placing parallelogram A B C D on parallelogram A′ B′ C′ D′, so that their sides A B, A′ B′ coincide, as angles A and A′ are equal, side A D falls on side A′ D′ and point D on D′. Side D C, parallel to A B, takes the direction of side D′ C′, parallel to A′ B′. For the same reason, side B C takes the direction B′ C′, point C becomes confounded with point C′, and the two parallelograms coincide.

Corollary.—*Two rectangles are equal when they have two adjacent sides equal, each to each.*

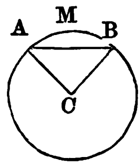
BOOK II.

DEFINITIONS.

33. The *circumference* is a plane line of which all the points are equally distant from one same point, situated in the middle and named centre.

34. The *circle* is the portion of a plane limited by the circumference.

35. The name of *radius* is given to every straight line drawn from the centre to the circumference. The radii of the same circumference are equal. A circumference is generally described in language by one of its radii.



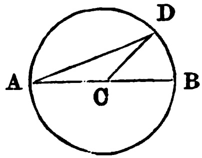
36. An *arc* of a circle is any part of a circumference; it has for its *chord*, or line *subtending* it, the straight line which joins its extremities.

Thus the straight line A B is the chord of the arc A M B of the circle C A. A chord belongs to two arcs, which united form the circumference. Only the smaller of the two arcs is generally considered.

37. The term *diameter* is applied to any chord which passes through the centre. All diameters in the same circle are equal, since each of them is twice a radius.

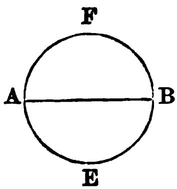
THEOREM I.

The diameter is the greatest of chords; it divides the circumference and the circle into two equal parts.



1. Let AD be a chord not passing through the centre. To the extremities of A D draw radii A C, C D. Then the straight line A D < A C + C D, or A D < A B.

COROLLARY. —Therefore the greatest straight line that can be inscribed in a circle is equal to its diameter.



2. The diameter A B divides the circumference A F E, and the circle bounded by it, into two equal parts.

For if A E B is folded over A B and placed on A F B, the curved line A E B must fall exactly on the curved line A F B; otherwise, in one or other of these semi-circumferences there would be points unequally distant from the centre, which is contrary to the definition of a circle.

THEOREM II.

A straight line cannot meet a circumference in more than two points.

For if it met it in three points, these three points would be equally distant from the centre. There would then be three equal straight lines drawn from the same point to the same straight line, which is impossible.*

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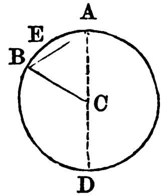
* Theorem XIV., Corollary II., Book I.

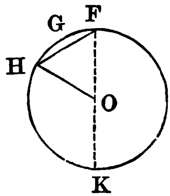
THEOREM III.

In the same circle, or in equal circles, equal arcs have equal chords. Reciprocally, *two arcs are equal if they have equal chords, and if they are both less or greater than a semi-circumference.*



Let circle C A be equal to circle O F, and arc A E B equal to arc F G H, then chords A B, F H of these arcs are equal.

Let these circles be superposed by placing the centre of one C on the centre of the other O, and point A on point F. Then the two circumferences coincide, and point B falls on point H, since the arcs A E B, F G H are equal by hypothesis. Therefore the chords A B, F H have the same extremities, and are equal.



Reciprocally. — Let arcs A E B, F G H be less than a semi-circumference, and subtended by equal chords A B, F H. Then they are equal.

For the radii C A, C B, O F, and O H, drawn to the extremities of the equal chords A B, F H, determine two triangles C A B, O F H, which have their three sides equal, each to each.* Consequently the angle C A B, opposite to the side C B, is equal to the angle O F H, opposite to the side O H. This being determined, apply the centre O of the circle O F on the centre C of the circle C A, and the point F on point A. Then the circumferences coincide, and the chord F H takes the direction A B, because of the equality of the angles O F H, C A B. Therefore point H falls on point B, and arc F G H is equal to arc A E B.

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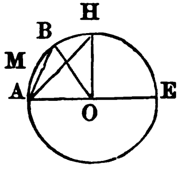
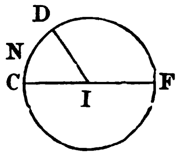
* Theorem IX., Book I.

THEOREM IV.

Of two unequal arcs, the greater is subtended by the greater chord.

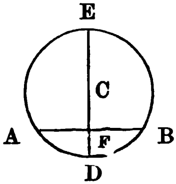




If in circles O A, I C, arc A M H is greater than arc C N D, chord A H will be greater than chord C D. For, with its origin at A, let us take in arc A M H an arc A M B equal to arc C N D. Then the chords A B, C D will be equal, and it remains to prove that chord A B is less than chord A H. But arc A M B being less than arc A M H, point B falls between the points A and H, and angle A O B is less than angle A O H. Consequently the two triangles A O B, A O H have an unequal angle included between two sides equal each to each, namely, O A common, and O B = O H, as radii of the same circle. Therefore (VII., 1st Book) the side A B, opposite to the angle A O B, is less than the side A H, opposite to the angle A O H.

THEOREM V.

The radius C D, *perpendicular to a chord* A B, *divides into two equal parts this chord and the arc* A D B *which it subtends.*



Fold the circle C D over the diameter D C E, applying the semicircle D A E to the semicircle D B E. The arc D A E coincides with the arc DBE, and the straight line F A takes the direction F B, because of the equality of the right angles C F A, C F B.

The point of intersection A of

the arc D A E and of the straight line F A falls, therefore, on the point of intersection B of the arc D B E and of the straight line F B. Therefore the straight line F A is equal to the straight line F B, and the arc D A to the arc D B.* Point F is therefore the middle of the chord A B, and point D the middle of the arc A D B.

COROLLARY I.—*The centre of a circle, the middle of a chord, and the middle of the arc subtended by this chord, are situated on the same straight line, perpendicular to the chord.*

As a straight line is determined by two points, or by a single point, on condition that the straight line is perpendicular to a given straight line, this corollary gives occasion to the six following enunciations :—

1. The radius, perpendicular to a chord, divides this chord and the arc which it subtends into two equal parts.

2. The perpendicular raised at the middle of a chord, and on that line, passes through the centre of the circle and the middle of the arc subtended by the chord.

3. The perpendicular, dropped from the middle of an arc on its chord, passes through the centre of the circle and the middle of the chord.

4. The radius, drawn through the middle of a chord, is perpendicular to it, and divides into two equal parts the arc which this chord subtends.

5. The radius, passing through the middle of an arc, divides the chord of that arc into two equal parts, and is perpendicular to it.

6. The straight line, drawn through the centres of an arc and of its chord, passes through the centre of the circle, and is perpendicular to the chord.

COROLLARY II.—The geometrical locus of the centres of the chords of a circle, parallel to a given straight line, is the diameter perpendicular to this line.

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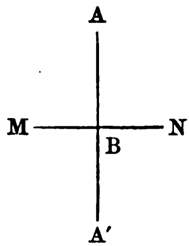
* Theorem III., Book II.

DEFINITIONS.

38. Two points A and A′ are called symmetrical in relation to a straight line M N, when the straight line M N is perpendicular to the centre of the straight line A A′. The straight line M N then takes the name of axis of symmetry.



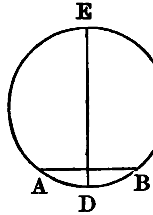
If the plane be folded over M N, and the upper part be brought down on the lower, the line B A will take the direction B A′, since the two angles M B A, M B A′ are right angles, and therefore equal. And as, moreover, B A = B A′, point A will coincide with point A′. Thus, when two points are symmetrical in relation to an axis, if one of the parts of a plane be turned over the axis to bring it down on the other part, the symmetrical points coincide.

39. Two curves, or two portions of the same curve, and more generally two figures, are styled symmetrical in relation to an axis, when the points of these two figures are two by two symmetrical in relation to this axis. It results from the preceding remark, that two figures, symmetrical in relation to an axis, can be applied one on the other; consequently they are equal.

40. A straight line is said to be the axis of symmetry of a curve when it divides that curve into two symmetrical parts.

It results from the foregoing propositions:—

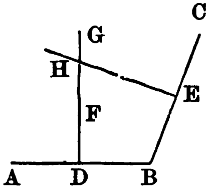
1. That every diameter is an axis of symmetry to the circumference.



2. That the diameter perpendicular to a chord is an axis of symmetry to each of the arcs which it subtends. For if by each of the points of the arc A B, for example, a chord be drawn perpendicular to the diameter D E, its centre will be on this diameter; that is, the arc B D is symmetrical to A D, or in other terms, the diameter D E divides the arc A D B into two symmetrical parts.

THEOREM VI.

Three points A, B, C *not in a straight line determine a circumference.*



Draw the straight lines A B, B C; then raise through the middle D, E of these lines the perpendicular D F on A B, and the perpendicular E G on B C. The straight lines D F, E G will meet, for they cannot be parallel, because the perpendiculars B A, B C, dropped from the same point B on these straight lines, do not coincide, by the hypothesis.

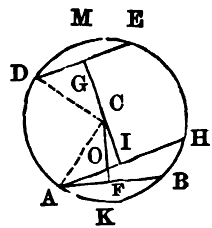
Let H be the intersection of these two lines. This point situated on the line D F, perpendicular to the middle of A B, is equally distant from the point A and B; it is also at the same distance from the two points B, C, as it is on the line E G, perpendicular to the middle of B C; it is therefore equally distant from the three points A, B, C. Moreover, it is the only point that enjoys this property; for every other point is outside at least one of the straight lines D F, E G, and therefore unequally distant from the points A, B, C.

The circumference described from the point H as centre with the radius A H passes, therefore, through the three points A, B, C; and it is the only one, since the point H is the only one equally distant from the three points A, B, C.

Corollary.—Two circumferences which have three points common coincide.

THEOREM VII.

Two equal chords are equally distant from the centre; and of two unequal chords, the lesser is the more remote from the centre.



1. Let the chord A B = D E; divide these chords in two by the perpendiculars C F, C G, and draw the radii C A, C D.

The right-angled triangles C A F, D C G have the hypothenuses C A, C D equal; moreover, the side A F, half of A B, is equal to the side D G, half of D E; therefore these triangles are equal,* and the third side C F is equal to the third C G; therefore, 1st, the two equal chords A B, D E are equally distant from the centre.

2. Let the chord A H be greater than D E; then arc A K H will be greater than the arc D M E.

On arc A K H take the part A K B = D M E. Draw the chord A B, and drop the perpendicular C F to that chord, and the perpendicular C I on A H. It is clear that C F is greater than C O, and C O greater than C I;† therefore, *à fortiori*, C F > C I. But C F = C G, since the chords A B, D E are equal. Therefore C G > C I; therefore of two unequal chords, the smaller is more remote from the centre.

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* Theorem XVI., Book I.

† Theorem XIV., Corollary I., Book I.

DEFINITIONS.

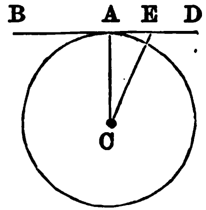
41. The name of *secant* of a circle is given to every straight line that has two points common to the circumference.

42. A line is *tangent* to a circumference when it has only one point in contact with that curve. This point is named the *point* of *contact.*

THEOREM VIII.

The perpendicular drawn to the extremity of a radius is a tangent to the circumference.

RECIPROCALLY.—*Every straight line tangent to a circumference is perpendicular to the radius at the point of contact.*



1. At the extremity A of the radius C A let the perpendicular B D be raised on that straight line; then it is tangent to the circumference C A.

For the distance C E from the centre C to any point E on the straight line B D, different to the point A, is greater than the radius C A perpendicular to B D.* Therefore point E is exterior to the circumference C A, and the line B D has only the point A common with that circumference.

RECIPROCALLY.—If the straight line B D touches the circumference C A at the point A, *it is perpendicular to the radius* C A.

For every point E on the straight line B D, different from A, being by hypothesis exterior to the circumference C A, the radius C A is the shortest line that can be drawn from the centre to the tangent B D; it is therefore perpendicular to that straight line.†

* Theorem XIV., Corollary I., Book I.

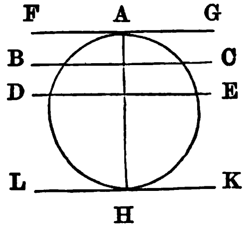
† Theorem XIV., Corollary I., Book I.

COROLLARY I.—Through a point on a circumference only one tangent can be drawn to that curve.

COROLLARY II.—The tangent is parallel to the chords, which the diameter, drawn to the point of contact, divides into two equal parts.*

THEOREM IX.

Two parallel straight lines intercept two equal arcs on a circumference.



The two parallels may be both secants or both tangents, or one secant and the other tangent. These three cases will be now examined.

1. If the two parallels are the secants B C, D E, the diameter A H, which is perpendicular to them, divides into two equal parts each of the arcs B A C, D A E subtended by these straight lines.* Therefore the arc A B is equal to the arc A C, and the arc A D equal to the arc A E; therefore the difference of the arcs A D, AB is equal to the difference of the arcs A E, A C; that is, arcs B D, C E, intercepted by the parallel secants B C, D E, are equal.

2. If one of the parallels is the secant B C, and the other the tangent F G, the radius drawn to the point of contact A is perpendicular to the tangent, and consequently to its parallel B C; * therefore it divides the arc B A C into two equal parts A B, A C.†

3. When the two parallels F G, K L are tangents, the diameter perpendicular to these two straight lines passes through their points of contact A and H; ‡ therefore the arc A B H is equal to the arc A C H.

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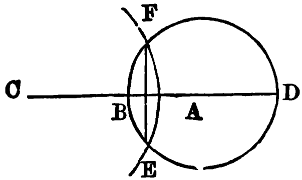
* Theor. XXII., Book I. † Theor. V., Corol. II., Book II.
‡ Theor. VIII., Corol. II., Book II.

DEFINITION.

43. Two circumferences are *tangents* at a common point, when they have the same tangent at that point.

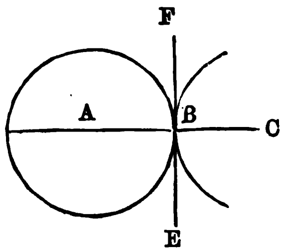
THEOREM X.

If two circumferences intersect, the straight line which unites their centres is perpendicular to the common chord, and divides it into two equal parts.



Let there be two circumferences A E, C E, which intersect at the points E and F; each of their centres A, C being equally distant from the two points E, F, the straight line A C is perpendicular to the common chord E F, and divides it into two equal parts.*

COROLLARY.—If the point E coincides with the point B, where the circumference A B cuts the straight line A C, the point F becomes also confounded with point B, since the straight line A C is perpendicular to the middle of E F. Then the two circumferences have only one common point B, and the straight line E F is tangent to both.



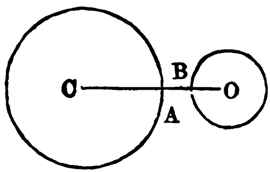
Therefore, 1st, *when two circumferences* A B, C B *have only one common point* B, *it is situated on the straight line* A C, *which joins their centres; 2nd, these circumferences have the same tangent* B F *at that point, that is, they are tangent to each other.*

* Theorem XV., Book I.

When two circumferences are drawn on the same plane, they have two points common, or only one, or none. In the two latter cases, one of the circumferences can be exterior or interior to the other; consequently these lines have in relation to each other only five different positions, to which the following five theorems correspond:—

THEOREM XI.

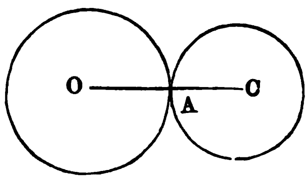
If two circumferences C A, O B, *having no common point, are exterior to each other, the distance of their centres* C, O *is greater than the sum of their radii* C A, O B.



The straight line C O, which joins the centres, cuts one of the circumferences at the point A, and the other at the point B; it is therefore equal to the sum of the radii C A, O B increased by the distance of the two points A and B, so that we have $CO > CA + OB$.

THEOREM XII.

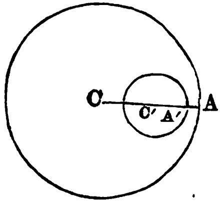
If two circumferences are tangents exteriorly, the distance of their centres is equal to the sum of their radii.



The point of contact A of the two circumferences is situated on the line of the centres, and then we have evidently $OC = OA + AC$.

THEOREM XIII.

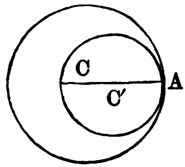
If two circumferences are interior, the distance of the centres is less than the difference of the radii.



For C C′ = C A — C′ A′ — A′ A, whence C C′ < C A — C′ A′

THEOREM XIV.

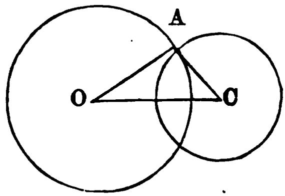
If two circumferences touch interiorly, the distance of the centres is equal to the difference of the radii.



For the point of contact A is on the line of the centres, giving C C′ = C A — C′ A.

THEOREM XV.

When two circumferences C A, O A *intersect, the distance of their centres* C, O, *is less than the sum of the radii* C A, O A, *and greater than their difference.*



Let A be one of the points of intersection of the two circumferences; this point being exterior to the straight line which joins the centres C and O,* the radii C A, O A make with the straight line C O a triangle in

* Theorem X., Book II.

which the side C O is at once less than the sum of the two other sides C A, O A, and greater than their difference.

Observation.—When two arcs of a circle intersect at a point A, it is said that they make an angle at that point, and this angle is measured by that formed by the tangents drawn to these arcs through the summit A in the direction of the same arcs.

Hence result the following theorems, which it is easy to prove :—

1. *The circumferences* C A, O A *intersect at the point* A *under the same angles as their radii* C A, O A *produced indefinitely.*

2. *The circumferences* C A, O A *intersect at the two points* A *and* B *under the same angles.*

DEFINITIONS.

44. To measure a magnitude is to find how many times it contains units of its kind and parts of units. When a magnitude is contained an exact number of times in two magnitudes of its kind, it is said to be their common measure.

45. Two magnitudes of the same kind are mutually *commensurable* or *incommensurable*, according as they have or have not a common measure.*

46. The ratio of the two magnitudes of the same kind is the number which would express the measure of the first, if the second were taken as unity.

THEOREM XVI. (*a*)

If two magnitudes of the same kind, A *and* B, *are mutually commensurable, their ratio is a whole or fractional*

* 1. Two magnitudes are called commensurable when they are multiples of a third magnitude, which is styled their common measure.

2. To measure a magnitude commensurable by unity is to seek how many of these unities or aliquot parts of unity are contained in it.

number, which is obtained by dividing the two numbers one by the other, and which expresses how many times these magnitudes contain their common measure M.

Let, for example, A = 25 M, and B = 8 M; then the common measure of M is $\frac{1}{8}$ of B; consequently A is equal to the $\frac{25}{8}$ of B, and the ratio of A and B is the fractional number $\frac{25}{8}$.

Reciprocally.—When the ratio of two magnitudes, A and B, *is a whole* or *fractional* number, *these magnitudes are mutually commensurable.* Thus, if the ratio of A to B is equal to $\frac{30}{7}$, the seventh of B is contained 30 times in A, and the magnitudes A, B have a common measure equal to the seventh of B.

When two magnitudes, A and B, are mutually incommensurable, it is impossible to measure the first by taking the second B as unit; but a magnitude A′ can be found, which is commensurable with B, and which differs from A as little as may be wished. For if you divide B into a very great number, say, a million equal parts, and if you take as A′ the greatest multiple of this fraction of B contained in A, the magnitude of A′ will not differ from A by a millionth part of B. In numerical applications, A′ is substituted for A, and when the relation of A′ to B is spoken of, the relation of A′ to B must be understood. Accordingly, to show that the relation of two magnitudes of the same kind, A and B, is equal to that of two other magnitudes of the same kind, C and D, the usual examination papers* direct the student only to consider values of those magnitudes which are commensurable together.†

* Abroad, particularly in France.

† Let it be conceived that unity is decomposed into any number n of parts equal to each other and less than an incommensurable magnitude G. Taking 1, 2, 3, 4 . . of these parts a series is formed (1) A_1, A_2, A, A_4, . . . A_k, A_{k+1}, . . : increasing beyond all limits, and respectively measured by the numbers $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n} \ldots \frac{k}{n}, \frac{k+1}{n}, \ldots$ Continuing far enough in series (1), two consecutive magnitudes will be found, A^k and A_{k+1}, which will comprehend the proposed magnitude G. Then substituting for G either A_k, or A_{k+1} an error will be

DEFINITIONS.

47. The term angle at the centre is given to an angle of which the summit is situated at the centre of a circle.

48. An angle is inscribed in a circle when it is formed by two chords which intersect on the circumference of that circle.

49. A *sector* is the portion of a circle included between two radii. A *segment* of a circle is the portion of a circle included between an arc and its chord.

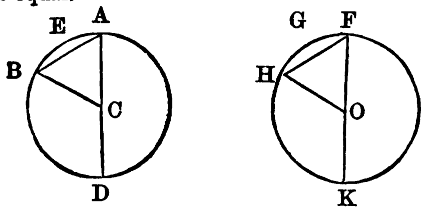
50. A *polygon* is inscribed in a circle when its summits are situated on the circumference.

RECIPROCALLY.—A circle is said to be circumscribed round a polygon.

THEOREM XVI. (*b*)

In the same circle, or in equal circles, equal arcs have equal chords. RECIPROCALLY, *two arcs are equal if they have equal chords, and if they both are larger or less than half a circumference.*

Let circle C A be equal to circle O F, and arc A E B equal to arc F G H; then the chords A B, F H of these arcs are equal.



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made less than the difference $A_{k+1} - A_k$, that is, as small as is wished, because this difference, which is the nth part of unity, can be diminished optionally, if n be taken large enough.

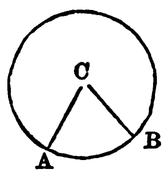
The result of operations to be effected on incommensurable numbers is the limit of the results obtained by substituting for each of them commensurable values approaching continually nearer to each other.

Let the two circles be superposed, placing centre C of the one on centre O of the other, and point A on point F. Then the two circumferences coincide, and point B falls on point H, because the arcs A E B, F G H are equal by hypothesis. Therefore chords A B, F H have the same extremities and are equal.

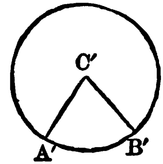
RECIPROCALLY.—Let arcs A E B, F G H be less than a semi-circumference, and subtended by equal chords A B, F H; then they are equal. For the radii C A, C B, O F, and O H, drawn to the extremities of the equal chords A B, F H, determine two triangles C A B, O F H, which have their three sides equal, each to each; therefore angle C A B, opposite to side C B, is equal to angle O F H, opposite to side O H. This being established, place centre O of circle O F on centre C of circle C A, and point F on point A; then the circumferences coincide, and chord F H takes the direction A B, on account of the equality of the angles O F H, C A B. Therefore point H falls on point B, and arc F G H is equal to arc A E B.

THEOREM XVII.

In the same circle, or in equal circles, two equal angles at the centre intercept equal arcs, and reciprocally.



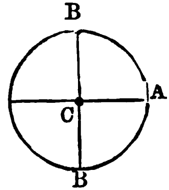
Let the circles C A, C′ A′ be equal; then arcs A B, A′ B′, intercepted by the two equal angles at the centre A C B, A′ C′ B′, are also equal.



To show this, superpose the two circles, placing centre C′ on centre C, and point A′ on point A. Then the radius C′ A′ coincides with the radius C A, and the circumference C′ A′ with the circumference C A. But the angle A′ C′ B′ being equal by hypothesis to angle A C B, radius C′ B′ falls on C B and coincides with it; therefore the arc A′ B′ coincides with the arc A B, and is equal to it.

Reciprocally.—If arcs A B, A′B′ are equal, the angles at the centre A C B, A′ C′ B′, which they intercept, are also equal.

For superposing circles C′ A′, C A so that the radii C′ A′, C A coincide, the arc A′ B′ coincides with its equal A B; consequently the radius C′ B′ takes the direction of the radius C B, and the angles at the centre A′ C′ B′, A C B, of which the sides coincide, are equal.



Corollary.—If from the summit of a right angle, A C B as centre, any circumference C A be described, the arc A B intercepted by this angle is equal to a quarter of the circumference.

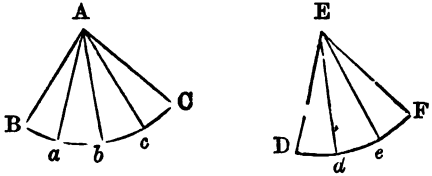
For the four angles at the centre formed by the straight lines C A, C B produced beyond point C are equal, as they are right angles; therefore they divide the circumference into four equal arcs.

Remark.—The fourth of the circumference is sometimes designated by the word *quadrant.*

THEOREM XVIII.

Two angles B A C, D E F *are in the same ratio* as the arcs* B C, D F *intercepted between their sides and described from their summits as centres with equal radii.*

Let us suppose that the common measure of the arcs B C, D F is contained, for instance, four times in B C and



* Book II., Definition 46.

three times in D F. Let us apply this common measure to the arcs, and let a, b, c, d, e, be the points of division. Joining these points to the respective centres, the angles B A a, etc., are obtained, all equal as they intercept equal arcs in equal circles.* The angle B A C contains four of these, whilst D E F contains three. Therefore the ratio of these two angles is equal to $\frac{4}{3}$; but the ratio of the arcs B C, D F is also equal to $\frac{4}{3}$. Therefore these two ratios are equal, and give the proportion B A C : D E F :: B C : D F.

Remark I.—As the angle varies by infinitely small degrees when the arc varies in that manner, it follows that this proportion remains if the arcs have no common measure.

Remark II.—Nothing prevents the supposition that one of the arcs is an entire circumference, and that the corresponding angle is therefore equal to four right angles.

Remark III.—The property demonstrated in this proposition is also expressed thus: An angle at the centre has for measure the arc included between its sides and described from its summit with an arbitrary radius.

Remark IV.—The right angle is generally taken as the angular unit, because the quarter of the circumference is taken as the unity of length for arcs.

THEOREM XIX.

The measure of an inscribed angle C A B *is equal to half the measure of the arc* C B *included between its sides.*

There are three cases of inscribed angles.

CASE I.—One of the sides A B of the inscribed angle is a diameter.

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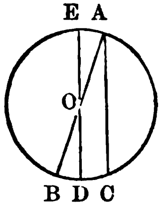
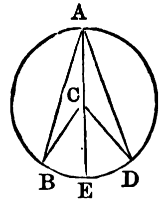
* Theorem XVII., Book II.



Through the point O draw the diameter D E parallel to A C. The two angles C A B, D O B are equal as corresponding angles formed by the parallels C A, D E cut by the secant A B. Therefore it is required to find the measure of the angle D O B.

But angles D O B, A O E are equal as opposed at the summit,* and intercept equal arcs. Therefore arc D B is equal to arc A E. The two arcs D B and C D, equal to the same arc A E, are equal to each other. Therefore arc D B is half of arc C B. But the angle at the centre D O B has the same measure as arc D B ;† therefore angle C A B, equal to D O B, has the same measure as arc D B, or half B C the arc contained between its sides.



Case II.—Let the centre of the circle be inside the angle B A D. Then drawing the diameter A E and the radii C B, C D, the angle B C E exterior to the triangle A B C is equal to the sum of the two interior angles C A B, A B C ;‡ but as the triangle B A C is isosceles, the angle C A B = A B C ; therefore the angle B C E is double of B A C. The angle B C E, as an angle at the centre, has for measure the arc B E ; therefore the angle B A C will have for measure the half of B E. For a like reason, the angle C A D will have for measure the half of E D ; therefore B A C + C A D, or B A D, will have for measure the half of B E + E D, or the half of B D.

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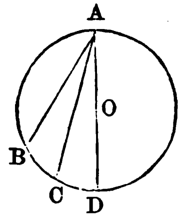
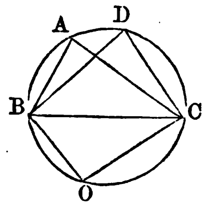
* Theorem IV., Book I. † Theorem XVII., Book II.
‡ Corollary I., Theorem XXVIII., Book I.



Case III.—The centre O falls outside the angle B A C. Draw the diameter A O D. Then the angle B A C is the difference of the angles B A D, C A D, which, according to the first case, have respectively as measure $\frac{1}{2}$ B D and $\frac{1}{2}$ C D; the difference of these arcs, that is, the half of arc B C, is therefore the measure of the proposed angle B A C.

Corollary I.—All angles B A C, B D C, etc., inscribed in the same segment are equal, for they have as measure the half of the same arc B O C.

Corollary II.—Every angle inscribed in a semi-circle is a right angle, for it has as measure half the semi-circumference, or a quadrant.

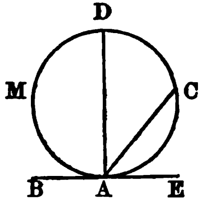


Corollary III.—Every angle B A C inscribed in a segment greater than a semi-circle is an acute angle, for it has for measure half the arc B O C, less than a semi-circumference; and every angle B O C inscribed in a segment less than a semi-circle is an obtuse angle, for it has for measure half the arc B A C, greater than a semi-circumference.

Corollary IV.—Every angle inscribed in one of the two segments determined by a chord is the supplement of any angle inscribed in the other segment.

THEOREM XX.

The angle B A C *formed by a tangent and a chord has for its measure half the arc* A M D C *included between its sides.*

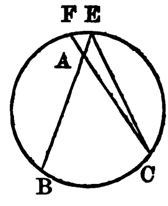


At the point of contact A draw the diameter A D. The angle B A D is a right angle, and has for measure half the semi-circumference A M D ;* the angle D A C has for measure the half of D C ; therefore B A D + D A C, or B A C, has for measure half A M D, plus the half of D C, or the half of the entire arc A M D C.

It could be proved in like manner that the angle C A E has for measure half the arc A C included between its sides.

THEOREM XXI.

The angle B A C *formed by the two secants* B E, A C, *and of which the summit is inside the circumference, has for measure half the arc contained between its sides, plus half the arc contained between the prolongations of the same sides.*



For the angle B A C exterior to the triangle A E C is equal to the sum of the angles A E C, A C E, which have respectively for measure half the arcs B C and F E.†

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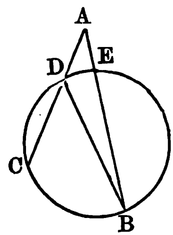
* Corollary II., Theorem XIX., Book II.

† Theorem XIX., Book II.

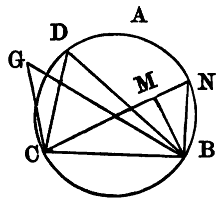
THEOREM XXII.

The angle B A C *formed by the two secants* A B, A C, *and of which the summit is exterior to the circumference, has for measure half the concave arc* B C, *minus half the convex arc* D E.



For the angle A is equal to the difference of the angles B D C, A B D, which have for their measure the first half of B C and the second half of D E.

The proposition is still true when one of the sides of the angle or both sides are tangents to the circumference, and the demonstration is the same.

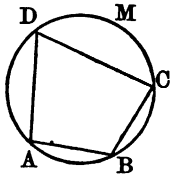


Corollary.—The arc B A C is the geometrical locus of the summits of the angles equal to C D B, and of which the sides pass through the points C and B. For, first, every angle inscribed in the segment B A C is equal to C D B; second, every angle C M B of which the summit is in the interior of the segment is greater than C D B. Thus, producing C M to the circumference at N, and joining N B, the angle C M B exterior to the triangle M N B is greater than the interior angle M N B; but the angle M N B = C D B, therefore C M B > C D B.

Third.—It may be seen in like manner that every angle C G B of which the summit is exterior to the segment is less than C D B.

THEOREM XXIII.

In every convex quadrilateral inscribed in a circle the opposite angles are supplementary.



For if the angles B and D be considered, then chord A C determines two segments A D C, A B C; and it has been seen that every angle D inscribed in the upper segment is the supplement of every angle B inscribed in the lower segment.*

Reciprocally.—If in a convex quadrilateral A B C D two opposite angles, B and D, are supplementary, the quadrilateral is *inscriptible;* † in other terms, the circumference determined by the three points A, B, C will pass by the fourth summit D.

For the summit D is a point situated above A C, and from which the chord A C will be viewed under an angle supplementary to B. But arc A M C is the locus of the points of the plane which enjoy this property.

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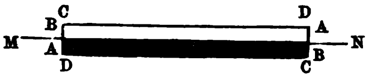
* Corollary IV., Theorem XIX.

† Admits of being inscribed.

PROBLEMS RELATING TO THE FIRST TWO BOOKS.

DEFINITIONS.

51. To draw a straight line on paper, an instrument is used, called a ruler. A ruler is a bar of wood or of metal, of which the faces are flat and the edges straight. When it is wished to draw the straight line determined by two given points on a plane, a ruler is placed on this plane, so that one of its edges passes through the two given points. Then the point of a pencil or pen is made to slide along the edge from one point to the other.



To be satisfied that the edge A B of a ruler A B C D is straight, a straight line M N is drawn on paper by making the point of the pencil slide along A B. Then it is tried to make the same edge A B coincide with this straight line but care is taken to place the extremity B to the left, which was before to the right, and reciprocally.

It is ascertained that this coincidence takes place—that is, that the edge of A B is straight—when, on tracing a second line along the edge A B, it is found that this straight line is confounded with the first. The ruler is well made if each of its edges satisfies this condition.

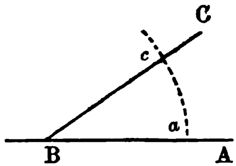
52. Compasses are an instrument with which a circumference is described on a plane. It is composed of two metal rods, commonly called the legs of the compasses. These legs terminate in a point at one of their extremities, and they are joined together at the other extremity

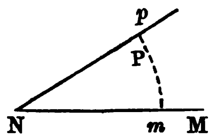
by a hinge, which allows the compasses to be opened more or less; that is to increase or diminish the angle which they form.

To describe a circumference of which the centre and the radius are given, the compasses are opened in such wise that the distance of its points may be equal to the radius; afterwards one of the points remains at the centre and the other point is made to turn round the centre; resting it on the paper. The movable limb is terminated by a pencil or pen, which describes the circumference.

PROBLEM I.

To make on a straight line M N *an angle having point* N *for summit, and equal to a given angle* A B C.





From the summit B of the angle A B C as centre, describe with any radius the arc *a c* between the sides of this angle; then from the common point N as centre, and with the same radius B *a*, describe an indefinite arc *p m* till it meets the straight line M N, and beginning from their intersection *m*, take with the compasses a portion *m* P of the arc *m p* equal to arc *a c*. Then draw the straight line N P. The angle M N P is equal to the angle A B C; for they are measured by the equal arcs *m p* and *a c*.*

PROBLEM II.

Two angles, A *and* B, *of a triangle being given, to find the third.*

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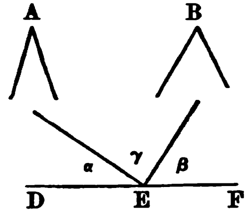
* Theorem XVII., Book II.

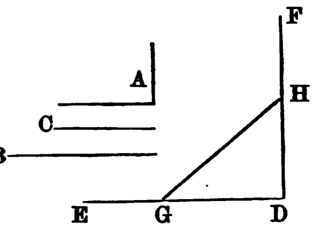
Draw the straight indefinite line D F. Make at point E the angle α = A, and the angle β = B;* the remaining angle γ will be the third angle required; for these angles taken together are equal to two right angles.



PROBLEM III.

When two sides, B *and* C, *of a triangle are given, and the angle* A *which they contain, to describe the triangle.*

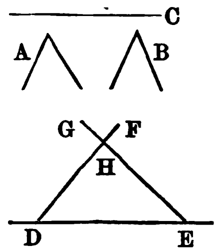
Having drawn the indefinite line D E, make at point D the angle E D F, equal to the given angle A. Take afterwards D G = B, D H = C, and draw G H. D G H will be the required triangle.†



PROBLEM IV.

When a side and two angles of a triangle are given, to describe the triangle.

The two given angles will be either both adjacent to the given side, or one adjacent and the other opposite to it. In the latter case, seek the third angle;‡ there will then be two adjacent angles. This being done, draw the straight line DE, equal to the given

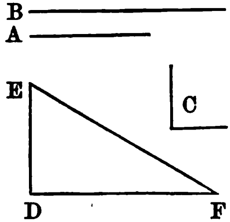


* Problem I., Book II. † Theorem V., Book I.
‡ Problem II.

side. Make at point D the angle E D F, equal to one of the adjacent angles, and at point E angle D E G, equal to the other; then the two lines D F, E G will cut at the point H, and D E H will be the triangle required.

PROBLEM V.

When two sides A B *of a triangle are given, with the angle* C *opposite to the side* B, *to describe the triangle.*

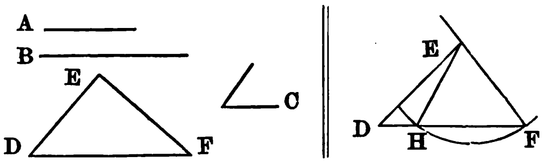


There are two cases :—

1. If the angle C is a right or an obtuse angle, make the angle E D F equal to the angle C ; take D E = A ; at point E as centre, and with a radius equal to the given side B, describe an arc cutting the line D F at F. Draw E F, and D E F will be the required triangle.

In this first case, the side B must be greater than A, for the angle C being a right or an obtuse angle is the greatest of the angles of the triangle.* Therefore the opposite side must also be the greatest.

2. If the angle C is acute, and the line B is less than A, the same construction as in case 1 takes place; then D E F is the required triangle. But if the angle C is acute, and side B is greater than A, then the arc described from the centre E, with the radius E F = A, will cut the side D F at two points, F and H, situated on



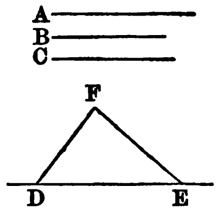
* Theorem XXVIII., Book I., Corollary II.

the same side of D ; therefore there will be two triangles D E F, D E H, which will equally satisfy the problem.

Scholium.—The problem would be impossible, in all cases, if the side B were smaller than the perpendicular dropped from E on the line D F.

PROBLEM VI.

The three sides A, B, C *of a triangle being given, to describe the triangle.*

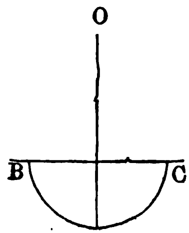


Draw D E, equal to the side A ; from point E as centre, and with a radius equal to the second side B, describe an arc. From point D as centre, and with a radius equal to the third side C, describe another arc that will cut the first at F. Draw D F, E F, and D E F will be the required triangle.

For the problem to be possible, it is necessary that the circumferences described, from the points D and F as centres, should intersect. For this to be possible, it is necessary that the side D E should be less than the sum of the two other sides, and greater than their difference.

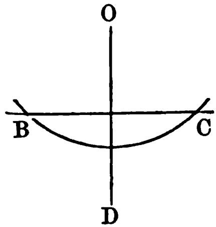
PROBLEM VII.

To draw from a given point O *a perpendicular to a given straight line* B C.



The point O may be on the straight line B C, or outside the line, but the perpendicular is constructed in the same way in both cases.

For let two points B, C be determined on the straight line B C, and



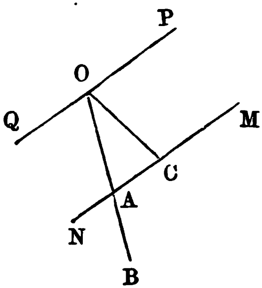
equally distant from point O, by describing from that point as centre an arc of a circle, which cuts the line B C.

From these two points B, C as centres, and with the same radius, which must be greater than half the distance B C, describe two arcs of a circle intersecting at point D, and then draw the straight line O D.

This line is perpendicular to the straight line B C, as it has two points O, D equally distant from the extremities of B C;* it is therefore the required perpendicular.

PROBLEM VIII.

To draw from a given point O *a parallel to a given straight line* M N.



1st Solution.— From the given point O draw a straight line O B, which cuts the given straight line M N at any point A, and make on O B the angle A O P, equal to the angle N A O.† The straight line Q P is parallel to M N; for these two lines make with the secant O A two equal alternate internal angles A O P, N A O.

2nd Solution.—Draw successively through point O the perpendicular O C on M N, and the perpendicular

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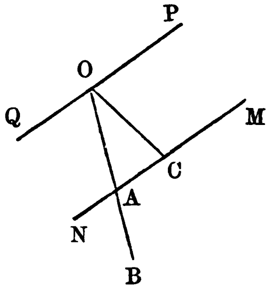
* Theorem XV., Book I. † Problem I., Book II.

O P on O C. The straight line O P is parallel to M N, for these two lines are both perpendicular to the same straight line O C.

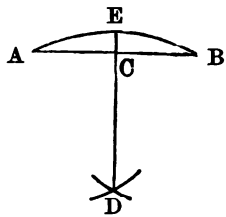
The operation is greatly abridged by making use of the instruments named square and protractor.



PROBLEM IX.

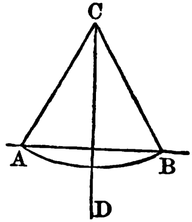
To divide a given arc or angle into two equal parts.

1. If it is wished to divide the arc A B in two equal parts, from the points A and B as centres, and with the same radius, describe two arcs intersecting at D. Through point D and the centre C draw C D, which will cut the arc A B in two equal parts at the point E.



For the two first points C and D are both equally distant from the extremities A and B of the chord A B; therefore the line C D is perpendicular on the middle of this chord; therefore it divides the arc A B in two equal points at the point E.

2. If it is necessary to divide into two equal parts the angle A C B, begin by describing from the summit C, as centre, the arc A B, and the rest of the construction as has been just said. It is evident that the line C D will divide the angle A C B into two equal parts.

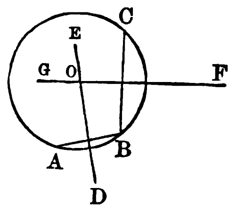


Scholium.—It is possible by the same construction to divide each of the halves A E, E B into two equal parts; and thus, by successive subdivisions, a given arc or angle can be divided into four, eight, sixteen and more equal parts.

PROBLEM X.

To find the centre of a given circle or arc.

Take, at option, in the circumference, or in the arc, three points A, B, C; join A B and B C, or imagine them joined. Divide these two lines into two equal parts by the perpendiculars D E, F G; then point O, where those perpendiculars meet, will be the required centre.*



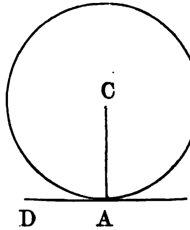
Scholium.—The same construction is used to make a circumference pass through three given points A, B, C, and also to describe a circumference in which the given triangle A B C shall be inscribed.

PROBLEM XI.

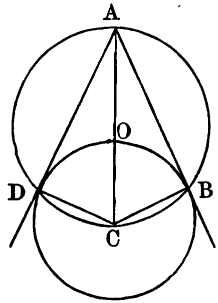
Through a given point to draw a tangent to a given circle.

If the given point A is on the circumference, draw the radius C A, and draw A D perpendicular to C A; A D will be the required tangent.

* Theorem VI., Book II., and Corollary, Theorem V., Book II.



If point A is outside the circle, join point A and the centre by the straight line C A; divide C A into two parts equally at point O. Then from point O as centre, and with the radius O C, describe a circumference, which will cut the given circumference at point B. Draw A B, and A B will be the required tangent.



For in drawing C B, the angle C B A, inscribed in a semicircle, is a right angle; therefore A B is perpendicular to the extremity of the radius C B; therefore it is a tangent.

Scholium.—The point A being out of the circle, it is perceived that there are always two equal tangents A B, A D, which pass through the point A. They are equal because the right-angled triangles C B A, C D A have the hypothenuse C A common, and the side C B = C D; therefore A D = A B, and angle C A D is also equal to C A B.

PROBLEM XII.

To inscribe a circle in a given triangle A B C.

Draw the bisectors* A O, B O of the angles A and B; these straight lines will intersect at a point O, which will be equally distant from the three sides AB, A C, BC.

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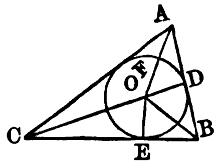
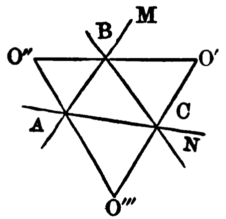
* The term *bisector* of an angle is given to the straight line that divides this angle into two equal parts.—See Theorem XVIII., Book I., *Note*.

If, therefore, from that point the perpendiculars O D, O F, O E be let drop on the sides of the triangle, these perpendiculars will be equal, and the circumference described from the point O as centre, with O D as radius, will be a tangent to the three sides.



Remark I.—Point O being equally distant from the sides B C, A C, belongs to the bisector of the angle C; therefore the three *bisectors of the angles of a triangle* meet in one and the same point.

Remark II.—If the bisectors of the two exterior angles M B C, B C N be drawn, the point of meeting O′ will be the centre of a circle tangent to the side B C, and to the prolongations of the other side.



In the same manner, the centres O″, O‴ of the two other circumferences tangent to one of the sides of the triangle and to the prolongations of the two others can be found.

Therefore there are in general four circumferences tangent to three given straight lines.

PROBLEM XIII.

To describe a tangent common to two circumferences.

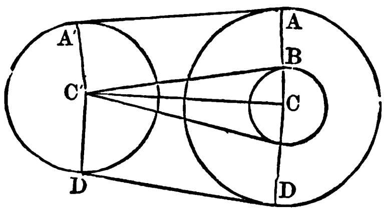
The circles can touch the same straight line on the same side, or on opposite sides of the line. In the first case the common tangent is exterior to the two circles; in the second case, it is interior.

1. Let us suppose the problem solved, and let A A′ be a common tangent, exterior to the two circumferences.

Let us draw the radii C A, C′ A′ to the points of contact, and the straight line C′ B parallel to A A′. The radii C A, C′ A′ being perpendicular to A A′, will be also perpendicular to the straight line C′ B; this latter line will therefore be a tangent to a circumference described from point C as centre, with a radius C B equal to C A − C′ A′.

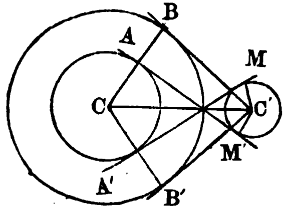
The following construction is deduced from the foregoing reasoning:—Describe a circumference from the point C′ as centre, with a radius equal to C A − C′ A′, and through the point C draw a tangent to this circumference. Knowing the point B, draw the line C B A, the line C′ A′ parallel to C A, and join A A′.

The construction shows that there are two solutions of the problem, since by the point C′ two tangents can be drawn to the circumference C B, and it is evident that the problem is only possible when C C′ $\geqq$ C A − C′ A′, or, in other terms, when the circumferences are not interior to each other.



2. Let it be proposed to draw a common tangent interior to two circumferences, of which the radii are C A and C′ M, and let A M′ be the line sought for. Then draw C A, C′ M′ to the points of contact, and the straight line C′ B perpendicular to A M′. The straight line A M′ being perpendicular to the radii C A, C′ M′, C B, will be perpendicular to the same straight lines; it will therefore be tangent to a circumference described from the point C as centre, with a radius C B equal to C A + A B, or C A + C′ M′.

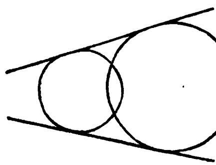
To solve the problem, a circumference will be described, having its centre at C, and of which the radius is the sum of the radii of the two given circumferences. A tangent C′ B will be drawn through C′ to this circumference, and the remainder of the construction will be completed as in the preceding case.



This problem has also two solutions; and it is only possible if $CC' \geqq CA + C'M$, that is, if the circumferences are exterior, or tangent externally.

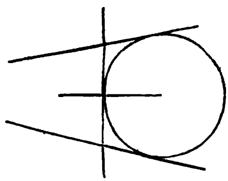
It follows from this (1), that two exterior and two interior tangents can be drawn to two circles that are exterior to each other.

(2.) Two circles that touch externally have two common exterior tangents and only one interior tangent, which is perpendicular to the straight line joining the centres of the circles.



(3.) If the circles intersect, they have only two common tangents, and these are exterior.

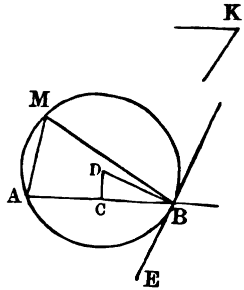
(4.) Two circles which touch internally have only one common tangent; it is exterior to the two circles and perpendicular to the straight line joining their centres.



(5.) Two circles interior to each other have no common tangent.

PROBLEM XIV.

To describe on a straight line A B *a segment of a circle, capable* of a given angle* K.



Make at the extremity B of the given straight line A B the angle A B E equal to the given angle K. Then raise the perpendicular C D in the middle of A B, and draw from point B the perpendicular B D on the line B E. The two lines C D, B D intersect at point D. From this point D, as centre, describe a circle with the radius D B; then the segment A M B, which is not on the same side of the straight line A B as the angle A B E, is the required segment. For the angle A M B, inscribed in this segment, has for measure the half of arc A B included between its sides.

But the given angle A B E, of which the side B E perpendicular to the radius B D is tangent to the circle, has also for its measure half of the same arc A B; † therefore the angle A M B is equal to the angle A B E, and the segment A M B is capable of the given angle.

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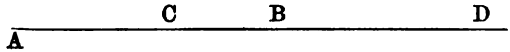
* This term—used in French treatises—explains itself, if traced to its Latin root, *capax*, holding, a segment capable of an angle = a segment holding an angle. Todhunter uses the term, "containing a given angle."—Prop. 34, Book III., p. 107, *School Euclid*, Ed. 1867.

† Theorem XIX., Book II.

BOOK III.

DEFINITIONS.

53. A straight line A B is said to be divided at point C into two parts A C, C B, proportional to two given



numbers, which can always be supposed to be whole numbers, such as 5 and 3, when the ratio of A C to C B is equal to the ratio of 5 to 3.*

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* This is the place to consider *harmonics*, recently introduced in French geometry.

Three numbers are said to form a harmonic proportion when the ratio of the excess of the first over the second to the excess of the second over the third is equal to the ratio of the first to the third. The second number has received the name of *harmonic* mean.

The term has resulted from the fact that to make a musical chord give out the three sounds *ut*, *mi*, *sol*, which form a perfect accord, three parts must be made to vibrate, proportional to the numbers 1, $\frac{4}{5}$, $\frac{2}{3}$, which gives rise to the harmonic proportion :—

$$\frac{1-\frac{4}{5}}{\frac{4}{5}-\frac{2}{3}}=\frac{1}{(\frac{2}{3}}$$

One of the most important of the harmonic proportions is the following : If three numbers *a*, *b*, *c* are in harmonic propor-

There is only one manner of dividing the line A B into two parts A C, C B, which are proportional to two given numbers 5 and 3. For to effect this division, it is necessary to divide A B into 5 + 3, or eight equal parts, and take for A C the first five eighths, starting from the extremity A, and for C B the three other eighths, that is, the remainder of A B.

The inquiry to find point C amounts to finding on the straight line A B a point of which the distances to points A and B are proportional to the numbers 5 and 3.

Presented in these terms, the question is susceptible of a second solution ; for if the distance A B is divided into 5 — 3, or two equal parts, and if on the part of A B produced, a length B D be taken equal to three times one of these parts, the straight line A D contains then 2 + 3, or five times the same part, and the ratio of the distances D A, D B from point D to the extremities A, B of the straight line BA is equal to that of the two numbers 5

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tion, the inverse $\frac{1}{b}$ of the harmonic mean b is equal to the arithmetical mean of the inverse quantities $\frac{1}{a}, \frac{1}{c}$ of the two other numbers a, c. For the proportion $\frac{a-b}{b-c} = \frac{a}{c}$ gives $a\,c - b\,c = a\,b - a\,c$, and $2\,a\,c = a\,b + b\,c$. Dividing both numbers of the last equality by $2\,a\,b\,c$, this gives $\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right)$; the demonstration of the theorem.

M. Chasles calls the *anharmonic* ratio of four points a, b, c, d, situated in a straight line $\underline{a \quad b \quad c \quad d}$ the number $\frac{a\,c}{b\,c} : \frac{a\,d}{b\,d}$, which is obtained by dividing the ratio of the distances of any one point, such as c, from two of the other points, for example a and b, by the ratio of the distances of the fourth point d from the two preceding points a and b.

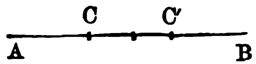
This subject of harmonics, giving a new treatment of questions related to Euclid's 5th Book, is reserved in France for superior geometry, and we only give this as a specimen of the new and interesting treatment of this question abroad.

and 3. Consequently point D satisfies the statement of the problem.

It is important to remark that point D is at the right of point A, because the given ratio is greater than unity, and it would be to the left if the ratio were less than unity. It is commonly said that points C and D divide the straight line A B into proportional segments, and these points are styled conjugate * points.

The points A and B divide *reciprocally* the straight line C D in two proportional segments; for from the equality $\frac{CA}{CB} = \frac{DA}{DB}$, the following is evidently deduced: $\frac{AC}{AD} = \frac{BC}{BD}$, which proves the reciprocity in the statement.

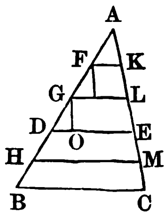
54. When a straight line A B is divided by a point C into parts proportional to two numbers, such as 5 and 3, if on A B a length A C′ equal to C B be taken, the distances C′ B, A C are equal to each other, and point C′ divides A B into two parts A C, C′ B, inversely proportional to the numbers 5 and 3, for their ratio $\frac{AC'}{BC'}$ is the inverse of the ratio $\frac{AC}{BC}$, which is equal to $\frac{5}{3}$.



* From the Latin *conjungo*, to join, connect, and *conjugo*, to join together and unite. *Points conjugués* (*conjugate points*) are terms adopted by Amiot in his *Eléments de Géométrie*, p. 84, and *Leçons Nouvelles*, part i., p. 142.

THEOREM I.

Every straight line parallel to one of the sides of a triangle divides the two other sides into proportional parts.



Draw the straight line D E parallel to the side B C of the triangle A B C; to prove that this line, which meets the two other sides A B, A C at the points D and E, divides them into proportional parts, let the ratio of A D to D B be supposed equal to $\frac{3}{2}$. The straight lines A D, D B have therefore a common measure A F, contained three times in A D and twice in B D.

Let F, G, D, H be the points which divide the side A B into 3 + 2, or five parts equal to A F.

Draw through these points the straight lines F K, G L, D E, H M, parallel to B C. Then these lines divide side A C into five equal parts.

For from one of the points of division of A B—for example, from point G—draw the straight line G O parallel to A C. The triangles A F K, D G O are equal, for their sides A F, G D are equal by hypothesis, and their angles F A K, D G O are equal as corresponding angles as well as A F K, G D O; consequently the sides A K, G O of these triangles are equal. But the quadrilateral G O E L is a parallelogram; therefore the side G O is equal to the side E L opposite to it, and the two divisions A K, E L of the straight line A C are equal.

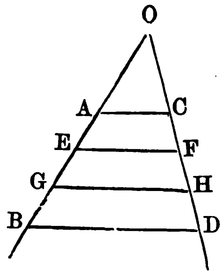
It could be proved in the same manner that A K is equal to each of the other divisions of A C. From this it follows that the length A K is a common measure of the two straight lines A E, E C, and that it is contained three times in A E, and twice in E C. Therefore the ratio of A E to E C is equal to $\frac{3}{2}$, or the ratio A D to D B.

COROLLARY I.—The ratio of the side A B to one of

its parts—for example, to A D—is equal to the ratio of the side A C to its parts A E, which corresponds to A D.

For, according to the preceding hypothesis, each of these two ratios is equal to $\frac{5}{8}$.

Corollary II.—The segments of two straight lines A B, C D determined by several parallels A C, E F, G H, B D, etc., are proportional.



For, let O be the point of intersection of the two straight lines A B, C D; in the triangle O E F, the line A C, being parallel to the base E F, will give the proportions $\frac{OE}{OF} = \frac{AE}{CF}$.

In the triangle O G H, a similar proportion will be determined: $\frac{OE}{OF} = \frac{GE}{FH}$.

Therefore, on account of the common ratio :—

$$\frac{AE}{CF} = \frac{GE}{FH}.$$

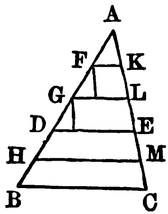
It could be proved in like manner that $\frac{GE}{FH} = \frac{BG}{HD}$; therefore, etc.

THEOREM II.

Every straight line D E *which divides two sides* A B, A C *of a triangle* A B C *into proportional parts, is parallel to the third side* B C.

Let D, E be the points where the straight line D E meets the sides A B, A C of the triangle A B C. Let the

ratio of A D to D B be supposed equal to that of A E to E C. Then the straight line D E is parallel to the third side B C of the triangle. For the parallel drawn through the point D to the straight line B C divides the side A C into two parts proportional to A D and D B; therefore this line passes through the point E, and coincides with D E, since there is only one manner of dividing A C, beginning from point A, into two segments proportional to A D and D B.

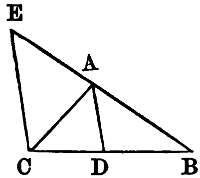


THEOREM III.

The bisector A D *of the angle* A *of the triangle* A B C *divides the base* B C *into two segments* B D, D C, *proportional to the two sides* A B *and* A C.

Through point C draw C E parallel to A D until it meets the prolongation of B A at E. In the triangle B C E, the line A D is parallel to the base C E. This gives the proportion:—

$$\frac{BD}{DC} = \frac{AB}{AE}.$$

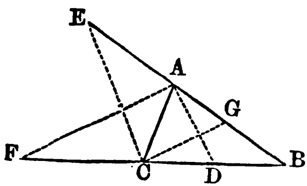


But the triangle A C E is isosceles; for because of the parallels A D, C E, the angle A C E = D A C, and the angle A E C = B A D. But by hypothesis D A C = D A B; therefore the angle A C E = A E C, and therefore A E = A C.

Therefore, substituting A C in the place of A E in the preceding proportion, this gives : $\frac{BD}{DC} = \frac{AB}{AC}$.

THEOREM IV.

The bisector of an angle exterior to a triangle cuts the opposite side at a point from which the distances to the extremities of this side are proportional to the adjacent sides.



Draw C G parallel to A F. In the triangle BAF, $\frac{BF}{FC} = \frac{AB}{AG}$.

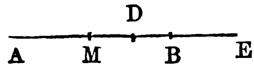
It can be shown, as in the last proposition, that the triangle A G C is isosceles; for angle A C G = F A C as alternate internal, and angle A G C = E A F as corresponding angles. But angles F A G and E A F are equal (hypothesis), therefore A G C = A C G. Then sides A G, A C are equal, and $\frac{BF}{FC} = \frac{AB}{AC}$.

Corollary.—If the point A moves in the plane so that the ratio A B to A C remains constantly equal to $\frac{m}{n}$, the bisectors of the angles B A C, E A C will pass always by the points D and F, and, since the ratios $\frac{BD}{DC} = \frac{BF}{CF}$, must remain equal to $\frac{m}{n}$.

Moreover, the straight lines A D, A F, bisectors of the two adjacent angles, are perpendicular to each other. Therefore the point A, in all its positions, will be on the circumference described on F D as diameter. Hence, *the geometrical locus of the points of which the distances to two points, B and C, are in given ratio, is the circumference of a circle.*

THEOREM V.

If a straight line A B *is divided into proportional segments by two points* D, E, *the half of this line is a mean proportional between the distance from its centre* M *to the two conjugate points* D, E, *and reciprocally.*



The hypothesis gives $\frac{AE}{AD} = \frac{BE}{BD}$. From this, by a known property of two ratios, may be deduced :

$$\frac{AE + BE}{AD + BD} = \frac{AE - BE}{AE - BD}.$$

Now, the sum A E + B E is equal to the double of M E, because point M is the centre of A B. In like manner, A D + B D is equal to twice M B, A E − B E is equal to 2 M B, and A D − B D is equal to 2 M E. Consequently, if the two terms of each ratio of the preceding equality be divided by two, this gives the new equality : $\frac{ME}{MB} = \frac{MB}{MD}$, which proves the statement of the theorem.

Reciprocally.—If the half of a straight line A B is a mean proportional between the distances from the middle M of this line to two points D, E, taken on its direction on the same side of point M, then points D, E divide the straighter line A B into proportional segments.

For, on applying to the equality given by the hypothesis $\frac{ME}{MB} = \frac{MB}{MD}$, the property of two equal ratios, previously employed, the result is : $\frac{AE}{AD} = \frac{BE}{BD}$.

Consequently, points D and E divide A B into proportional segments.

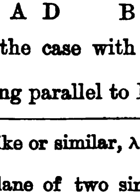
DEFINITIONS.

55. Two triangles are called similar when they have equal angles and their homologous* sides proportional. The term homologous sides means those sides opposite the equal angles, each to each. Similar polygons are those which have their angles equal, each to each, and their homologous sides proportional; the term homologous sides applying to those which are adjacent to the equal angles.

56. The term ratio of *similitude* of two polygons is given to the constant ratio of two homologous sides. When this ratio is equal to unity, the two polygons are equal; for they can be made to coincide by superposition, since they have all their parts (sides and angles) equal, each to each, and disposed in the same order.

THEOREM VI.

Cutting a triangle by a parallel to one of its sides, a second triangle is determined, similar to the first.



Let the straight line D E be parallel to the side B C of the triangle A B C. Then the triangle A D E is similar to the triangle A B C.

For these triangles have the angle A common; their angles A B C, A D E are equal as corresponding angles in relation to the parallels B C, D E, and the same is the case with the two angles A C B, A E D.

Moreover, the straight line D E being parallel to B C,

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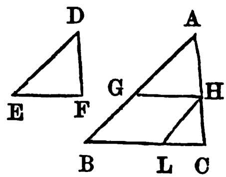
* Homologous, from the Greek ὁμοῖος, like or similar, λόγος, word or reason.

Two straight lines situated in the plane of two similar polygons are called *homologous*, when their extremities are two by two, homologous points; such are, for example, the diagonals in relation to homologous summits or vertices.

the ratio of A D to A B is equal to that of A E to A C.* To prove that this latter ratio is equal to that of D E to B C, draw from E the straight line E F, parallel to the side A B. This line E F divides the two other sides A C, B C into proportional segments, and the ratio of A E to A C is equal the ratio B F to B C, or that of D E to B C, since the lines B F and D E are equal as opposite sides of the parallelogram B D E F. The triangles A D E, A B C have therefore their angles equal, each to each, and their homologous sides proportional; that is, they are similar.

THEOREM VII.

Two equiangular triangles have the homologous sides proportional.



Let A B C, D E F be two triangles having their angles equal, each to each, namely, A = D, B = E, C = F. Then the homologous sides will be proportional, thus giving $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

Take A G = D E, A H = D F, and join G H; the triangles A G H, D E F will be equal, as having an equal angle included between equal sides. Therefore the angle A G H will be equal to the angle E. But angle E = B, therefore angle B = A G H; therefore G H is parallel to B C, and this gives the proportion: $\frac{AB}{AG} = \frac{AC}{AH}$.

Next, draw H L parallel to A B; giving the proportion: $\frac{AC}{AH} = \frac{BC}{BL}$, or $\frac{AC}{AH} = \frac{BC}{GH}$; for the straight lines B L, G H are equal as parallels included between parallels.†

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* Theorem I., Book III.

† Theorem XXXI., Corollary I., Book I.

Comparing the last proportion to the previous one, it can be deduced by the common ratio, that :—

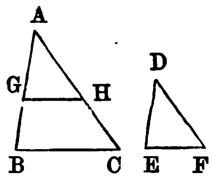
$$\frac{AB}{AG} = \frac{AC}{AH} = \frac{BC}{GH}, \text{ or } \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

COROLLARY.—For two triangles to be similar, it is sufficient that they have two angles equal, each to each, for then the third will be equal in both, and the two triangles will be equi-angular.

THEOREM VIII.

Two triangles which have an equal angle included between proportional sides are similar.

Let the angle A = D, and let it be supposed that $\frac{AB}{DE} = \frac{AC}{DF}$.



Then the triangle A B C is similar to D E F.

Take A G = D E, and draw G H parallel to B C. Then angle A G H will be equal to A B C,* and triangle A G H will be equi-angular with triangle A B C.

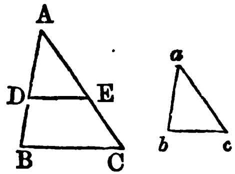
This gives : $\frac{AB}{AG} = \frac{AC}{AH}$.

But by the hypothesis $\frac{AB}{DE} = \frac{AC}{DF}$, and by the construction A G = D E ; therefore A H = D F. The two triangles A G H, D E F have therefore an equal angle included between equal sides ; therefore they are equal. But triangle A G H is similar to A B C ; therefore D E F is also similar to A B C.

* Theorem XXIII. Book I.

THEOREM IX.

Two triangles which have their sides proportional are equi-angular.



Let A B C, $a b c$ be two triangles which have the sides A B, A C, B C, respectively proportional to the sides $a b$, $a c$, $b c$; then these triangles are similar.

Take on A B a length equal to $a b$, and draw through point D the straight line D E parallel to B C. Triangles A D E, A B C are similar,* and their homologous sides are proportional. But the sides $a b c$ are proportional, by hypothesis, to the sides of the triangle A B C; therefore they are also proportional to those of triangle A D E; that is, the following equalities may be deduced :—

$$\frac{ab}{AD} = \frac{ac}{AE} = \frac{bc}{DE}.$$

But A D is equal to $a b$; consequently A E is equal to $a c$, and D E equal to $b c$. The triangles A D E, $a b c$, which have therefore the three sides equal, each to each, are equal, and the triangle $a b c$ is similar to the triangle A B C.

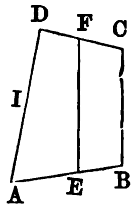
Corollary.—The three cases of similitude of two triangles proved in the three preceding theorems, correspond with the three cases of equality enunciated in Book I., Propositions V., VI., and IX.

Scholium I.—The two latter propositions show that in the triangles, the equality of the angles is a consequence of the proportionality of the sides, and reciprocally, so that one of the conditions is sufficient to secure the similitude of the triangles. This is not the case with figures having more than three sides, for even in the case

* Theorem VI., Book III.

of quadrilaterals, without changing the angles, the proportion of the sides can be changed, or without altering the sides, the angles can be changed; thus the proportionality of the sides cannot be a consequence of the equality of the angle, nor *vice versâ.*

It can be seen, for example, that by drawing E F parallel to B C, the angles of the quadrilateral A E F D are equal to those of the quadrilateral A B C D, but the proportion of the sides is different; in like manner, without changing the four sides A B, B C, C D, point B can be brought near or removed from D, without changing the angles.



Scholium II.—The two foregoing propositions, which properly only form one, taken with that of the square on the hypothenuse, are the most important and productive propositions in geometry; they suffice almost by themselves for all applications, and the solution of all problems; and this is because all figures can be divided into triangles, and every triangle can be divided into two right-angled triangles. Thus the general properties of triangles contain implicitly those of all figures.

THEOREM X.

Two triangles which have their sides parallel or perpendicular, each to each, are similar.

For, let A, B, C be the angles of one triangle, A′, B′, C′ the angles of the other triangle. It is known that two angles which have their sides parallel or perpendicular are equal or supplementary.*

* Theorems XXVI. and XXVII., Book I.

Therefore only one of these three hypotheses can be made.

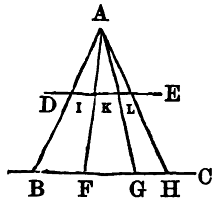
1.	A+A′ = 2 right angles.	B+B′ = 2 right angles.	C+C′ = 2 right angles.
2.	A+A′ = 2 „	B+B′ = 2 „	C=C′
3.	A = A′	B = B′ and thence	C = C′

But in the first hypothesis the sum of the angles would be equal to six right angles. In the second hypothesis, this sum would be superior to four right angles. Therefore the third is the only admissible hypothesis; therefore the triangles are equi-angular and similar.

Remark.—The homologous sides of the two triangles are the parallel or perpendicular sides.

THEOREM XI.

The lines AF, AG, *drawn as may be wished through the summit of a triangle, divide the base* BC *and its parallel* DE *proportionally, so as to give:* $\frac{DI}{BF} = \frac{IK}{FG} = \frac{KL}{GH}$, etc.



For, since DI is parallel to BF, the triangle ADI is equi-angular with ABF; and this gives the proportion :—

$$\frac{DI}{BF} = \frac{AI}{AF}.$$

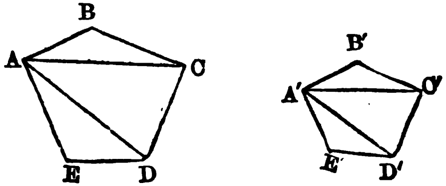
In the same manner, IK being parallel to FG, it follows that $\frac{AI}{AF} = \frac{IK}{FG}$; therefore, on account of the common ratio, $\frac{DI}{BF} = \frac{IK}{FG}$.

It would be found in like manner that $\frac{IK}{FG} = \frac{KL}{GH}$ etc.; therefore the line DE is divided at the points I, K L, as the base BC is divided at the points F, G, H.

THEOREM XII.

Two similar polygons can be decomposed into the same number of triangles, similar each to each, and similarly placed.

Let there be two similar polygons A B C D E, A′ B′ C′ D′ E′. From their homologous summits draw the homologous diagonals A C, A′ C′, A D, A′ D′, etc. These lines decompose the polygons into the same number of triangles similarly placed. Then these triangles are similar, two by two.

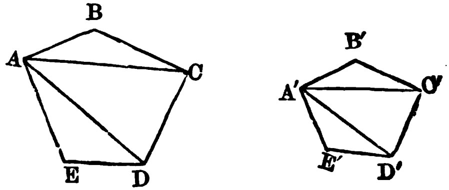


1. The triangles A B C, A′ B′ C′ have their angles B, B′ equal by hypothesis, and included between proportional sides; for it results from the similitude of the two polygons that the ratio of A B to A′ B′ is equal to that of B C to B′ C′. Therefore these two triangles A B C and A′ B′ C′ are similar.

2. The triangles A C D, A′ C′ D′ are also similar, because they have an equal angle included between proportional sides. For the angle A C D is the difference of the two angles B C D, B C A. But angle B C D is equal to B′ C′ D′ by hypothesis, and angle B C A is equal to B′ C′ A′, because they are homologues in the similar triangles A B C, A′ B′ C′; therefore angle A C D is equal to the difference of the two angles B′ C′ D′, B′ C′ A′, that is, to angle A′ C′ D′.

Moreover, the ratio of A C to A′ C′ is equal to the ratio of B C to B′ C′, because of the similarity of the triangles A B C, A′ B′ C′; and the ratio of B C to B′ C′ is equal, by hypothesis, to that of C D to C′ D′; therefore

the sides A C, A′ C′ are proportional to the sides C D, C′ D′. Consequently the triangles A C D, A′ C′ D′ are similar.



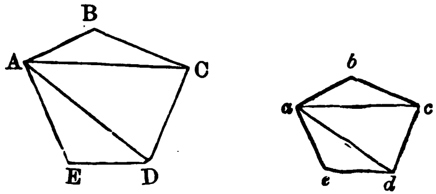
3. It would be possible to prove in like manner the similitude of the other triangles. Therefore the polygons A B C D E, A′ B′ C′ D′ E′ are decomposed into the same number of triangles, similar each to each, and similarly placed.

Corollary.—*The homologous diagonals of two similar polygons are proportional to the homologous sides.*

THEOREM XIII.

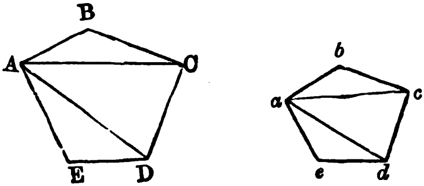
Reciprocally, two polygons A B C D E, *a b c d e, composed (as has been previously seen) of the same number of similar triangles, similarly placed, have their angles equal, each to each, and their homologous sides proportional; and consequently they are similar.*

For the similitude of the triangles A B C, *a b c* gives the angle A B C = *a b c*, and the angle B C A = *b c a;* the



similitude of the triangles A C D, *a c d* gives the angle A C D = *a c d*, from which it is concluded that B C D =

b c d, and so on. Moreover, on account of the similitude of the same triangle, the following series of equalities

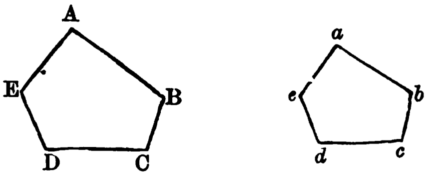


will result: $\frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac} = \frac{CD}{cd} = \frac{AD}{ad} = \frac{DE}{de} = \frac{AE}{ae}$. Therefore the two polygons have also their homologous sides proportional, and are similar.

THEOREM XIV.

*The perimeters * of two similar polygons are proportional to the homologous sides.*

Let A B C D E, *a b c d e*, be two similar polygons. Then, by the very definition of similar polygons, the following



equalities ensue:—$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \frac{DE}{de} = \frac{EA}{ea}$.

But when there is a succession of equal ratios, the ratio of the sum of the numerators to the sum of the

* Perimeter, a word of Greek derivation, meaning outline: περί, around, μέτρον, measure.

Perimeter is incorrectly identified with circumference. The former applies to rectilinear, the latter to curvilinear figures.

denominators is equal to any of the given ratios. Therefore :—

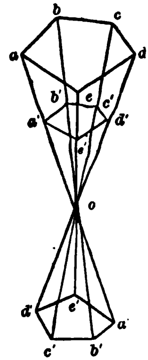
$$\frac{AB + BC + CD + DE + EA}{ab + bc + cd + de + ea} = \frac{AB}{ab}.$$

THEOREM XV.

If any point o be joined to the summit of a polygon a b c d e, and if on the straight lines o a, o b, o c . . or their prolongations, points a′, b′, c′, be taken, so that,

$$\frac{o\,a'}{o\,a} = \frac{o\,b'}{o\,b} = \frac{o\,c'}{o\,c} = \ldots = r,$$

r being a given number; then the polygon a′ b′ c′ d′ e′ will be similar to polygon a b c d e.



Suppose, first, that points a', b', c' are on the straight lines $o\,a$, $o\,b$, $o\,c$; then triangles $o\,a\,b$, $o\,a'\,b'$, having an equal angle between two proportional sides, are similar, and their homologous angles $o\,a\,b$, $o\,a'\,b'$ are equal. Now, these angles are corresponding in relation to the two straight lines $a\,b$, $a'\,b'$, and to the secant $o\,a$; therefore the straight line $a'\,b'$ is parallel to $a\,b$ and in the same direction; and this gives : $\frac{a'\,b'}{a\,b} = \frac{o\,a'}{o\,a} = r.$

It could be proved in like manner that the straight lines $b'\,c'$, $c'\,d'$, etc., are respectively parallel to the straight lines $b\,c$, $c\,d$, etc., and in the same direction, and that $\frac{b'\,c'}{b\,c} = r$, $\frac{c'\,d'}{c\,d} = r$, etc.

The angles $a\,b\,c$, $a'\,b'\,c'$ are equal because they have their sides parallel and directed, two and two, in the same direction; the same thing is the case with the

angles $b\,c\,d$, $b'\,c'\,d'$, etc. Therefore the polygons $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$, which have their angles equal, each to each, and their homologous sides proportional, are similar.

If the points a', b', c' . . . were on the prolongations of the straight lines $o\,a$, $o\,b$, $o\,c$ . . . beyond the point o, the similitude of the polygons $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$ could be proved in like manner; but their homologous sides would be parallel, and directed in an opposite direction, instead of in the same direction.

Remark.—When the points a', b', c' are situated on the straight lines $o\,a$, $o\,b$, $o\,c$ . . . the polygons $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$ are said to be *similar and similarly placed;* these latter polygons now described are, on the contrary, similar and inversely placed, if the points $a'\,b'\,c'$ . . . be beyond the point o.

M. Chasles * gave to this similitude of form and of position the name of *homothetie,*† direct in the former case, and inverse in the second case. The point o is the *centre of similitude,* and the straight lines $o\,a$, $o\,a'$ are the *radii vectors* ‡ of the homologous points a, a', etc. If the *ratio of similitude* r is made to vary from o to ∞ and the position of the centre of similitude, all the polygons homothetic to the given polygon $a\,b\,c\,d\,e$ will be obtained. It is important to remark that when two polygons $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$ are inversely homothetic, they can be made *direct* by causing one of them—for instance, $a'\,b'\,c'\,d'\,e'$—to turn in its plane round the centre of similitude o of an angle of 180°; for the radii vectors $o\,a'$, $o\,b'$, $o\,c'$ . . . are then applied on the homologous radii $o\,a$, $o\,b$, $o\,c$.

* Professor of Superior Geometry at the Collége de France, and one of the first geometers of the present age.

† Homothetie, from ὅμοιος, like, θέσις, position.

‡ Radii vectors or vectores are the straight lines drawn from the two foci of an ellipse to any one point of the circumference of an ellipse. In the symmetrical polygons points a, a' correspond to the foci.

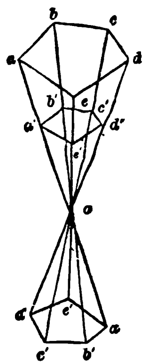
The distances from o to the symmetrical polygons will be constant.

The reasoning is fully developed in Conic Sections, the *courbes usuelles* of French mathematicians.

THEOREM XVI.

Reciprocally, if two similar polygons have their homologous sides parallel, the straight lines drawn through their homologous summits meet in the same point, and the two polygons are homothetic.

Let $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$, be two similar polygons; let it be supposed that their homologous sides are parallel and drawn in the same direction. Then draw the straight lines $a\,a'$, $b\,b'$, which meet at point o, and it is inferred that the straight lines $c\,c'$, $d\,d'$ . . . pass through that point. To prove this, let point o be joined successively to the two points c, c'. Then triangles $o\,a\,b$, $o\,a'\,b'$ are similar, on account of the parallelism of the two straight lines $a\,b$, $a'\,b'$.* Triangles $o\,b\,c$, $o\,b'\,c'$ are also similar, because they have an equal angle included between two proportional sides.† For angle $o\,b\,c$ is the difference of the two angles $a\,b\,c$, $a\,b\,o$; but angle $a\,b\,c$ is equal to angle $a'\,b'\,c'$ by hypothesis, and the angle $a\,b\,o$ is equal to $a'\,b'\,o$, because they are homologues in the similar triangles $o\,a\,b$, $o\,a'\,b'$; therefore the angle $o\,b\,c$ is equal to the difference of the angles $a'\,b'\,c'$, $a'\,b'\,o$—that is to say, the angle $o\,b'\,c'$.



Moreover the ratio of $b\,o$ to $b'\,o$ is equal to that of $a\,b$ to $a'\,b'$, on account of the similitude of the two triangles $o\,a\,b$, $o\,a'\,b'$, and the ratio of $a\,b$ to $a'\,b'$ is equal by hypothesis to that of $b\,c$ to $b'\,c'$; therefore the sides $b\,o$, $b'\,o'$ are proportional to the sides $b\,c$, $b'\,c'$; consequently

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*The orem VI., Book III. † Theorem VIII., Book III.

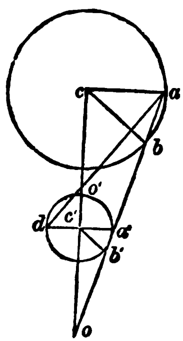
the triangles $o\,b\,c$, $o\,b'\,c'$, are similar, and the angle $b\,o\,c$ is equal to its homologue $b'\,o\,c'$; the straight line $o\,c'$ coincides, therefore, with the straight line $o\,c$—that is, the straight line $c\,c'$ passes by the point o. It could be proved in like manner that the straight line $d\,d'$ passes by that point, etc.

The demonstration would be the same, if the homologous sides of the two polygons were turned in opposite directions.

THEOREM XVII.

In two circles, 1st, the straight lines which join the extremities of parallel radii in the same direction pass through the same point, named direct centre of similitude.

2nd, the straight lines which join the extremities of parallel radii turned in opposite directions pass also through the same point, named inverse centre of similitude.



1. Let c, c' be the centres of two circles. Then draw in the same direction the parallel radii $c\,a$, $c'\,a'$, and draw the straight line $a\,a'$, of which the prolongation cuts the straight line $c\,c'$ at point o. The triangles $o\,a\,c$, $o\,a'\,c'$ are similar, and give $\dfrac{o\,c}{o\,c'} = \dfrac{c\,a}{c'a'}$.

The position of point o on the straight line $c\,c'$ only depending on the ratio of the radii $c\,a$, $c'\,a'$, and not at all on their direction, it is inferred that straight lines, such as $a\,a'$, which join the extremity of parallel straight lines in the same direction, meet at a point o; this point is named the direct *centre of similitude* in the two circles $c\,a$, $c'\,a'$.

2. Let the parallel radii $c\,a$, $c\,d'$ be drawn in opposite directions, and the straight line $a\,d'$ meeting $c\,c'$ at point o'. From the similitude of the triangles $o'a\,c$, $o'd'c'$ results the proportion: $\frac{o'c}{o'c'} = \frac{c\,a}{c'd'}$.

Consequently, the position of point o only depends on the ratio of the radii of the two circles, and the straight lines which join the extremities of parallel radii, directed in opposite directions, meet at this point, which is named *inverse centre of similitude.*

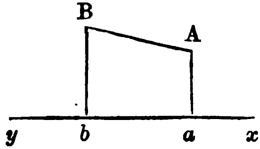
Remark I.—The construction of the two centres of similitude results from the preceding demonstration.

Remark II.—The previous reasonings can be applied, without modification, to two figures containing curved lines.

Remark III.—Architects make great use of symmetrical figures. The façade of a building is generally composed of two parts, symmetrical in relation to the vertical line of this façade. The doors, the windows, the cornice, the portions of the roof, pediments, trussed beams, etc., are also generally composed of symmetrical parts. The ornaments used in the decoration of buildings, such as socles, volutes, and beadings, are also symmetrical figures.

Remark IV.—Printing is nothing else than an industrial means of making a figure equal by symmetry to another figure. Thus the figure formed by the upper surfaces of the printing type is equal by symmetry to the impression which they leave on the paper. The same thing occurs in the drawing engraved on a plate of copper or steel, or traced on a lithographic stone, and with the drawing obtained on paper by printing. This is the reason why the draughtsman on wood, the engraver, and the lithographer must form the lines from right to left, in order that in the symmetrical impression obtained on paper the words may be arranged from left to right. It is also the reason why the type and the letters inscribed on plates for printing are reversed.

DEFINITIONS.



57. The term projection of a point A on an indefinite straight line $x\,y$ is given to the foot a of the perpendicular drawn from point A on that line.

If from the extremities A and C of a straight line A B perpendiculars be dropped on an indefinite straight line $x\,y$, the distance $a\,b$ from the projections of A and B is the *projection of the line* A B on $x\,y$.

58. To simplify the enunciations of the following theorems, the term *product of two lines* will be given to the product of two numbers which express the magnitude of those lines, measured with the same unity. The term *square* of a line will be given to the second power of the number which expresses the measure of that line.

59. When four numbers A, B, C, D are such that the ratio $\frac{A}{B}$ of the first to the second is equal to the ratio $\frac{C}{D}$ of the third to the fourth, it is said that the last D of these numbers is a *fourth proportional* to the three others A, B, C.

If two mean numbers C, B are equal, the equality $\frac{A}{B} = \frac{C}{D}$ becomes: $\frac{A}{B} = \frac{B}{D}$, and the fourth term D takes the name of *third proportional* to the two numbers A, B. In this case the mean number B is the *mean proportional* between the numbers A and D. It results from the preceding equality that $B^2 = A \times D$.

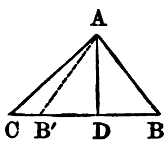
Therefore the mean proportional B between the numbers A and D is equal to the square root of the product of those numbers.

60. Two straight lines drawn between the sides of an angle, or of the angle vertically opposite, are *anti-paral-*

Reciprocally.—If through point B, taken on one of the sides of angle B A C, two straight lines B C, B E be drawn in the interior of the angle, such that $AB^2 = AC \cdot AE$, these two straight lines are anti-parallel in relation to that angle.

THEOREM XIX.

If from the summit A *of the right angle of a right-angled triangle* A B C, *the perpendicular* A D *be dropped on the hypothenuse,* 1st, *each side of the right angle is a mean proportional between the hypothenuse and its projection on the hypothenuse;* 2nd, *the perpendicular* A D *is a mean proportional between the two segments* B D *and* C D *of the hypothenuse.*



For, first, the straight lines A C, A D are anti-parallels in relation to angle B, because they form a right angle, one with side B A, the other with side B C. This gives: $BA^2 = BC \cdot BD$.

The relation $CA^2 = BC \cdot CD$ could be proved in like manner.

2. The straight lines A C and A D being anti-parallels in relation to angle B, the angles B A D and C are equal. Therefore, if the triangle A D B be applied to triangle A D B′, by turning it over A D the angle B′ A D will be equal to C, and the two straight lines A B′ and A C will be anti-parallel in relation to angle A D C. This will give $AD^2 = B'D \cdot CD$, or $AD^2 = BD \cdot CD$.

THEOREM XX.

The three sides of a right-angled triangle being valued in numbers, by means of a common unit, the square of the number which measures the hypothenuse is equal to the sum of the squares of the numbers which measure the two sides of the right angle. More briefly—

In every right-angled triangle, the square on the hypothenuse is equal to the sum of the squares on the two other sides.

For, adding the two ratios (Diagram, Theorem XIX):

$$AB^2 = BC \cdot BD, \; AC^2 = BC \cdot CD,$$

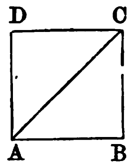
the result is—

$$AB^2 + AC^2 = BC\,(BD + CD), \text{ or } AB^2 + AC^2 = BC^2.$$

Corollaries.—By this theorem one of the sides of a right-angled triangle can be calculated when the other sides are known.

If the sides b and c of a right angle are known, the hypothenuse a results from the formula—

$$a^2 = b^2 + c^2 \text{ whence } a = \sqrt{b^2 + c^2}.$$



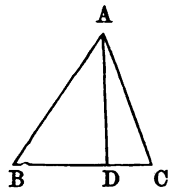
If the hypothenuse a and one of the sides of the right angle b be known, the other side c is found by the formula, $c^2 = a^2 - b^2$, whence $c = \sqrt{a^2 - b^2}$.

It results, moreover, from the foregoing proposition, that the *ratio of the diagonal* of a square to the side of that square is equal to $\sqrt{2}$. The diagonal AC is, in fact, the hypothenuse of a right-angled isosceles triangle ABC, in which $AC^2 = AB^2 + BC^2 = 2\,AB^2$; whence $\frac{AC^2}{AB^2} = 2$, and $\frac{AC}{AB} = \sqrt{2}$.

Thus the diagonal and the side of a square are two straight lines mutually *incommensurable*, since their ratio is an incommensurable number.

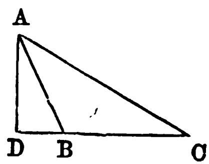
THEOREM XXI.

In every triangle, the square on a side opposite to an acute angle is equal to the sum of the squares on the two other sides, minus twice the rectangle of one of these sides by the projection of the second on the first.



Let C be an acute angle in the triangle A B C. Dropping a perpendicular A D on B C, it determines the result: $AB^2 = AC^2 + BC^2 - 2BC \times CD$.

There are two cases: 1st, if the perpendicular falls within the triangle A B C, this will give $BD = BC - CD$, and consequently $BD^2 = BC^2 + CD^2 - 2BC \cdot CD$. Adding on both sides AD^2 and observing that the right-angled triangles A B D, A D C give $AD^2 + BD^2 = AB^2$, and $AD^2 + DC^2 = AC^2$, the result will be $AB^2 = BC^2 + AC^2 - 2BC \times CD$.

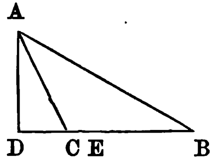


Case 2.—If the perpendicular A D falls outside the triangle A B C, this will give $BD = CD - BC$, and consequently $BD^2 = CD^2 + BC^2 - 2CD \times BC$.

Adding to both sides AD^2, the result is the same as in Case 1: $AB^2 = BC^2 + AC^2 - 2BC \times CD$.

THEOREM XXII.

In every obtuse-angled triangle, the square of the side opposite to the obtuse angle is equal to the sum of the squares on the two other sides, plus twice the rectangle formed by one of the sides multiplied by the projection of the second on the first.



Let A B be the side opposite to the obtuse angle C in the triangle A B C. Drawing a perpendicular A D to B C, produced, it will give $AB^2 = AC^2 + BC^2 + 2BC \times CD$. The perpendicular cannot fall within the triangle; for if it fell, for example, on E, the triangle A C E would have at the same time the right angle E and the obtuse angle C, which is impossible; therefore it falls out side the triangle and gives $BD = BC + CD$.

From this it follows that $BD^2 = BC^2 + CD^2 + 2BC \times CD$. Adding AD^2 to both sides, and making the reductions as in Theorem XXI., the conclusion is $AB^2 = BC^2 + AC^2 + 2BC \times CD$.

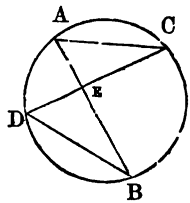
Scholium.—The right-angled triangle is the only one in which the sum of the squares on the two sides is equal to the square on the third side. If the angle included by these sides is acute, the sum of their squares will be greater than the square on the opposite side; if it is obtuse, it will be less.

THEOREM XXIII.

If from a point A taken in the plane of a circle, secants are drawn, the product of the distances from this point to the two intersections of each secant with the circumference is constant.

Point A may be inside or outside the circle. In both cases the demonstration is the same.

CASE I. — Let point E be inside the circle. Draw chords A C and B D. Then the angles A C D, A B D inscribed in the same segment are equal, and therefore the chords A C, B D are anti-parallels in relation to the angle A E D. Therefore this ratio gives E A . E B = E C . E D.



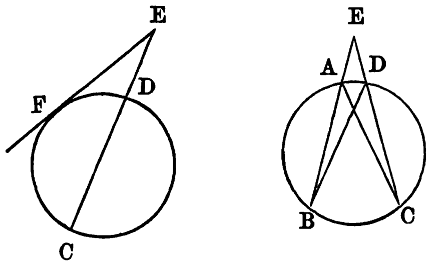
RECIPROCALLY.—When two straight lines A B, C D, being produced, meet at apoint E, such that it gives the ratio E A . E B = E C . E D, the extremities A, B, C, D are situated in the same circumference.

For the given ratio proves that the two straight lines A C and B D are anti-parallels in relation to the angle A E D. Consequently the angles A C D, D B A are equal; and if on the straight line A D there is described a segment capable of the angle A C D, the circumference, which passes through the three points A, C, D, will also contain point B.

THEOREM XXIV.

If through a point outside a circle, a secant and a tangent be drawn to that circle, the tangent is a mean proportional between the entire secant and its exterior part.

Case II.—Let E be the exterior point, as in the last figure,* and let it be supposed that the secant E D C



turns round point E till it coincides with the tangent E F. Then the entire secant E C and its exterior part E D will become both equal to the length E F of the tangent, and the relation E A . E B = E C . E D, taken at its limit, will be the following : E A . E B = EF^2.

Moreover, the demonstration of the general case can be applied directly to this particular case.

Thus, angles E B F, A F E, one inscribed and the other formed by a tangent and a secant, have both for measure half the arc A F. (See diagram, p. 104.)

The equality of these angles proves the anti-parallelism of the straight lines A F and B F in relation to angle E, and therefore leads to the relation EF^2 = E A . E B.

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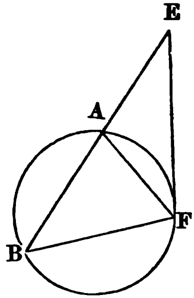
* Theorem XXIII., Book III.



RECIPROCALLY.—If three points A, B, F, situated, the two former, A and B, on one side of angle E, and the third, F, on the other side, are such as to give the relation $EF^2 = EA \cdot EB$, the circumference which passes by these three points is tangent at F to side E F.

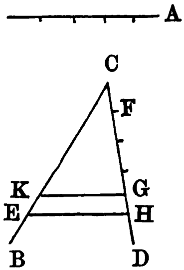
For, by (1), the given relation proves the anti-parallelism of the straight lines A F and B F with regard to the angle E. The angles A F E, E B F are therefore equal; consequently, if a circumference be described passing by A and F, and tangent at F to the straight line E F, this circumference, by the well-known construction of the capable angle,* will pass by point B.

* Problem XIV., Book II.

PROBLEMS ON BOOK III.

PROBLEM I.

To divide a straight line into a certain number of equal parts.



Let it be required to divide the straight line A into five equal parts. Take on the side B C of any angle B C D the length C E equal to A, and carry five times following on the other side C D an arbitrary length C F.

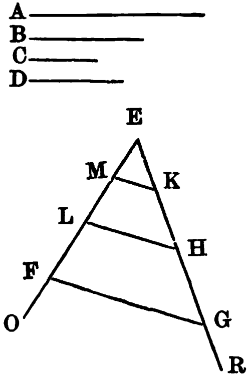
Let G be the fourth point of division and H the fifth. Then draw the straight line E H, to which, through point G, draw the parallel G K.

The straight line G K divides the two sides C E, C H of the triangle C E H into proportional segments, since it is parallel to the third side. But the segment G H is, by hypothesis, a fifth of the side C H; therefore the segment E K is also a fifth of the side C E, or of the straight line A.

To divide A into five equal parts, it is therefore sufficient to carry the length E K five times following on that line.

PROBLEM II.

To divide a straight line A *into parts proportional to given lengths* B, C, D.

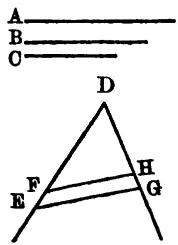


On the side E O of any angle O E R, let the length E F be taken equal to A, and on the other side E R, the lengths E K, K H, H G, equal respectively to the given lines B, C, D. Then let the straight line F G be drawn, to which parallels H L, K M are drawn through points H and K.

These parallels divide the straight line E F,* or A, into segmênts proportional to the lines E K, K H, and H G; that is to say, to the given lines B, C, and D.

PROBLEM III.

To construct a fourth proportional to three given straight lines A, B, C.



On the side D E of any angle E D G let the lengths D E, D F be taken, equal respectively to the given lines A and B, and on the other side the length D G, equal to C. Then draw the straight line E G, to which let the parallel F H be drawn.

The straight line D H is the fourth proportional re-

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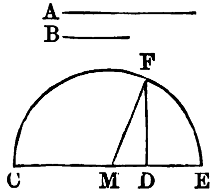
* Theorem I., Book III.

quired; for F H being parallel to E G, it follows that $\frac{DE}{DF} = \frac{DG}{DH}$*; that is to say, $\frac{A}{B} = \frac{C}{DH}$.

Remark.—If the lines B and C were equal, D H would be the third proportional to the two lines A and B.

PROBLEM IV.

To construct the mean proportional between two given lines, A *and* B.



Take on an indefinite straight line the lengths C D, D E, equal respectively to the given lines A and B. Then describe a semi-circumference on C E as diameter. Next raise at point D a perpendicular to C E, and prolong it to point F, where it meets the circumference.

This perpendicular is the line required, for it is a mean proportional between the two segments C D, D E of the diameter C E; that is, between the two given lines A and B.

Remark.—The mean proportional D F between two unequal lines C D, D E is less than half their sum M F.

PROBLEM V.

To construct the two conjugate points† *which divide a straight line* A B *into segments proportional to two given lengths, m and n.*

At the extremity A of A B draw any straight line A C equal to m, and through the other extremity

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* Theorem II., Book III. † See Definition 53, Book III.

BOOK V.

AREAS OF POLYGONS AND OF THE CIRCLE.

§ I.—Areas of Polygons.

DEFINITIONS.

64. The term area expresses the extension of a limited surface.

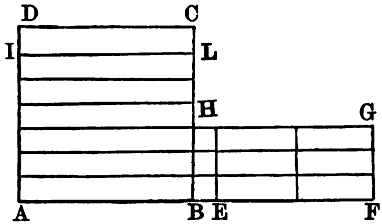
65. To measure areas, it is usual to take a square as unity. It has been already stated that surfaces are measured indirectly.

66. Two areas are *equivalent* when they have an equal measure and cannot coincide; they are equal when they can coincide.

THEOREM I.

The ratio of two rectangles A B C D, E F G H, *having equal bases* A B *and* E F, *is the same as that of their height* A D *and* E H.

There are two cases; for it may happen that the heights A D, E H are commensurable, or they may be incommensurable.



1. If the heights A D and E H are commensurable, let D I be their common measure, and let it be assumed that it is contained 7 times in the height A D, and 3 times in the height E H; the proportion of A D to E H is then $\frac{7}{3}$.

Drawing through the points of division, parallels to the bases, the rectangle A B C D will be divided into 7 partial rectangles, and rectangle E F G H into 3 partial rectangles. All these partial rectangles are equal, because they have equal bases * and equal heights.† Therefore, each of them is a common measure of the two rectangles A B C D and E F G H. Consequently, the proportion of the two rectangles A B C D and E F G H is also $\frac{7}{3}$,—the same as that of their height.

2. If the heights are incommensurable, let one of them, for example E H, be supposed to be divided into equal parts, and applying one of these parts to the height A D from the point A, the last point of division cannot fall upon D, because the heights are incommensurable. Let I be the last point of division, and let the rectangle A B L I be constructed. As the heights A I, E H are commensurable, we shall have, according to Case 1, $\frac{ABLI}{EFGH} = \frac{AI}{EH}$. As the parts into which E H is supposed to be divided may be as small as may be desired, point I can approach D as closely as is wished; therefore the rectangle A B C D is the limit of the variable rect-

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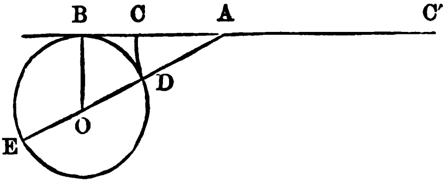
* Hypothesis. † Parallels comprised between parallels.

This secant and its exterior part E F are the two lines required. For their difference F G, is equal to B C, and their product is equal to BE^2, or to A^2.*

PROBLEM VIII.

To divide a straight line A B *in mean and extreme ratio, that is, to divide it into two parts* A C, B C, *such that the greater* A C *is a mean proportional between the other part* B C *and the whole line* A B.

Let it be supposed that the problem is solved. Let C be the required point. Then: $\frac{AB}{AC} = \frac{AC}{BC}$.



Applying to this equality a well-known property of equal ratios, it is inferred that $\frac{AB + AC}{AC + BC} = \frac{AB}{AC}$.

But the straight line A B is equal to the sum of the lines A C, B C; therefore $(AB + AC)\,AC = AB^2$. This new equality shows that to have point C, two straight lines must be constructed, A B + A C, A C, of which the difference is equal to A B, and the product is equal to AB^2.

Then on A B, starting from point A, a length must be taken equal to the less of these two lines. Hence is obtained this construction: At point B on the straight line

* Theorem XXIV., Book III.

AB, let the perpendicular BO be raised, equal to the half of AB. From point O as centre, let a circumference be described with radius BO, and let the secant AE be drawn through point A and the centre of the circle. The entire secant AE and its exterior part AD are the two lines, of which the difference is equal to AB and the product equal to AB^2.*

Then, taking on AB a length AC equal to the line AD, the point C will be determined, dividing AB into mean and extreme ratio.

COROLLARY.—If the length of the straight line AB be designated by a, the right-angled triangle AOB gives: $AO^2 = AB^2 + BO^2$. Whence, $AO^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$.

But $AC = AO - BO$; therefore $AC = \frac{a\sqrt{5}}{2} - \frac{a}{2} = \frac{a(\sqrt{5}-1)}{2}$; the other segment of the straight line $= \frac{a(3-\sqrt{5})}{2}$.

Remark.—The preceding problem is only a particular case of the following one: *On the straight line which passes through two given points* A *and* B, *to find a point* C, *such that its distance to point* A *is a mean proportional between its distances to point* B *and the line* AB.

It is evident that point C, given by the previous construction, satisfies the conditions of this enunciation.

But in one of the prolongations of the straight line AB there is another point having the same property. This point is not to the left of point B, because its distance from point A cannot be at the same time greater than its distance to point B and also greater than AB. Therefore it must be sought to the right of A.

* Theorem XXIV., Book III.

$\frac{R}{c} = \frac{a \times b}{S^2}$; that is, $\frac{a \times b}{S^2}$ is the measure of the rectangle R.

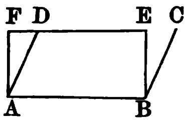
Therefore the area of a rectangle is equal to the product of its base by its height, divided by the second power of the side of the square, taken for unity; the three straight lines being measured with the same arbitrary unity.

If the side of the square which is taken for unity of surface is the linear unity, then $S^2 = 1$; and consequently, $\frac{R}{c} = a \times b$, which is the enunciation of the theorem.

Corollary.—*The area of a square is equal to the second power of its side,* because the square is a rectangle of which the base and height are equal.

THEOREM IV.

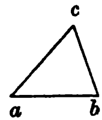
The area of a parallelogram A B C D *is equal to the product of its base by its height.*



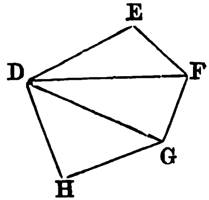
At points A and B, raise the perpendiculars A F, B E on A B, till they meet the side D C or D C produced; then the right-angled triangles A D F and B E C, having the hypothenuses A D, B C equal, as also the sides A F and B E, are equal.* Taking away from the trapezium A B C F the triangle B E C, there remains the rectangle A B E F; and taking away from the same trapezium the triangle A D F, there remains the parallelogram A B C D. But if you take away equal quantities from the same quantity, the remainders are equal; therefore the rectangle A B E F is equivalent to the parallelogram A B C D.†

* Theorem VI., Book I. † Definition 15.

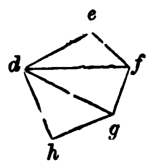
Then the triangle *a b c* is similar to the triangle A B C * because they have two angles equal, each to each.



2. Let it be proposed to construct on the straight line *d e* a polygon similar to the polygon D E F G H. Let it be supposed that this line is homologous to the side D E, and let the polygon D E F G H be decomposed into triangles by drawing the diagonals D F, D G from the summit D.



Then on *d e* let the triangle *d e f* be constructed similar to the triangle D E F, and afterwards, on *d f*, triangle *d f g* similar to triangle D F G, and lastly, on *d g*, triangle *d g h* similar to triangle D G H.



Then the polygon *d e f g h* is similar to the polygon D E F G H, for they are composed of the same number of similar triangles similarly placed. †

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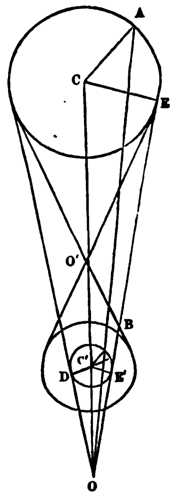
* Theorem VIII., Book III. † Theorem XIII., Book III.

PROBLEM X.

To draw a tangent common to two circles.

Determine the centres of similitude O, O′ of the two circles, by dividing the distance of their centres into segments proportional to the radii C A, C′ B.

Next draw through both points O, O′ the tangents to the circle C.



These straight lines will also touch the other circle C′; for the points E, E′ at which one of these straight lines—for example, tangent O E—meets the two circumferences being homologous, the radii C E, C′E′ are parallel, and the straight line O E, which is perpendicular on C E by hypothesis, is also perpendicular on C′E′. Therefore the two circles C, C′ have the same tangent.

The same demonstration could be used for the other straight lines.

Scholium.—This problem has four solutions when the two circles are exterior to each other, because the two centres of similitude O, O′ are in that case exterior to circle C. There are two solutions when the circles touch externally, two when they are secants, and only one if they touch internally. These results are identical with those previously determined.*

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* Theorems XII., XIV. and XV., Book II.

BOOK IV.

REGULAR POLYGONS AND THE CIRCUMFERENCE OF THE CIRCLE.

DEFINITIONS.

61. The name of regular polygon is given to every polygon, convex or concave, which has its sides equal and its angles equal. The equilateral triangle and the square are regular polygons.

A polygon is *inscribed* in a circle when all its summits are on the circumference of that circle. Reciprocally, it is said that the circle is *circumscribed* round the polygon.

62. A polygon is *circumscribed* or *described* * round a circle when all its sides are tangents to the circumference of that circle. It is said then that the circle is *inscribed* in the polygon.

63. The term *limit* of a variable magnitude is given to a fixed magnitude, to which the variable magnitude can approach indefinitely without equalling it.

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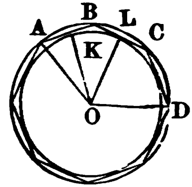
* The term *described* is retained in our English editions of Euclid. The French Geometricians always employ the term *circumscribed*, which is more exact.

THEOREM I.

Every regular polygon A B C D *can be inscribed in a circle and described round it.*



1. It may be affirmed that the circumference drawn through three consecutive summits A, B, C passes through the following summit D.

Through the centres K and L of the sides A B, B C of the polygon, let perpendiculars be raised on A B, B C. These perpendiculars intersect at point O, which is the centre of the circumference determined by the three summits A, B, and C.

To prove that the straight line O D is equal to the radius O A of that circumference, let the two quadrilaterals O L B A, O L C D be superposed, by folding the figure over the straight line O L. Then, as the right angles O L B, O L C are equal, the side L C takes the direction L B, and point C is applied to B, because L is the middle of the side B C. In like manner, angles L C D, L B A of the regular polygon being equal, as well as the sides C D, B A, the straight line C D takes the direction B A, and point D falls on A. But the straight lines O D, O A, of which the extremities coincide, are equal; therefore the circumference described from point O as centre, with the radius O A, passes by the summit D.

It could be proved in the same manner that this circumference passes through the other summits of the polygon A B C D . . . ; therefore this regular polygon can be inscribed in a circle.

2. The sides A B, B C, etc., of the polygon A B C being equal chords of the circumscribed circle, the perpendiculars O K, O L, etc., dropped from the centre O on these chords, are also equal.* Therefore the circum-

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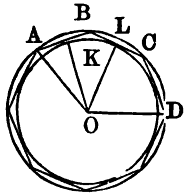
* Theorem VII., Book II.

ference described from point O as centre, with the radius O K, passes by the middle K, L, etc., of the sides A B, B C, etc., and is a tangent to each of those lines. Consequently the regular polygon A B C can be described round a circle.



Remark.—Point O, which is at the same time the centre of the inscribed and circumscribed circles, is named the *centre* of the regular polygon.

The term *radius* or *apotheme* of the regular polygon is given to the radius of the circumscribed, and also to that of the inscribed, circles.

The term *angle at the centre* of the regular polygon is given to the angle of two consecutive radii O A, O B. The angles at the centre are equal, because they intercept equal arcs in the circumference circumscribed.* The ratio of each of these angles to the right angle is therefore equal to $\left(\frac{4}{n}\right)$, *n* being the number of sides of the regular polygon supposed to be convex.

THEOREM II.

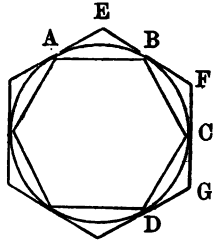
If a circumference be divided into any number of equal arcs A B, B C, C D, *etc.—1st, the chords of these arcs form a regular convex polygon, inscribed in the circumference.*

2nd. The tangents, drawn through the points of division, form also a regular convex polygon, described round the circumference.

1. The arcs A B, B C, C D, etc., being equal,† their

* Theorem XVII., Book II. † Theorem XVI. (*b*), Book II.

chords are equal and the inscribed polygon A B C . . . has its sides equal.



Then its angles are also equal; for the inscribed angle A B C has for its measure* half the sum of the arcs A D, D C included between its sides. The inscribed angle B C D has in like manner for measure half the sum of the arcs B A, A D included between its sides. But arcs D C and B A are equal; therefore the angle A B C is equal to the angle B C D. It can be proved in like manner that the other angles of the inscribed polygon A B C D are equal, and therefore the polygon is regular.

2. The polygon E F G, formed by the tangents drawn to the circumference through the points A, B, C, etc., is also regular; for, in the first place, each of the triangles E A B, F B C, etc., is isosceles, because the tangents drawn from the same exterior point to the same circle are equal;† then these triangles are equal, because they have an equal side adjacent to two equal angles.

For example, take triangle E A B, G C D; their sides A B, C D are equal, in consequence of the hypothesis. The angle E A B, which has for its measure half the arc A B,‡ is equal to the angle G C D, measured by half the arc C D. The same remark applies to the angles E B A, G D C.

Therefore the triangles E A B, G C D are equal.

Hence results, 1st, the equality of the angles E, F, G, etc., of the polygon; 2nd, the equality of the tangents E B, F B, F C, G C, etc., and therefore the equality of the sides E F, F G, etc., of the polygon, which is consequently regular.

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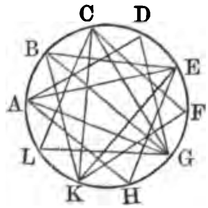
* Theorem XIX., Book II. † Problem XI., Book II., Scholium.
‡ Theorem XIX., Book II.



COROLLARY.—When a circumference is divided into a certain number of equal parts, A B, B C, etc.—for instance, into ten—and, starting from A, if the points of division from n to n be united by straight lines, n being a whole number less than the half of ten, a regular concave polygon of ten sides is formed if n is a prime number to 10. But when the numbers n and 10 are not prime to each other, and d is their common measure, the regular polygon, thus constructed, has only $\frac{10}{d}$ sides.

Let n be first assumed equal to 3, and consequently prime to 10. Then let the ten points A, B, C, D, etc., which divide the circumference into ten equal parts, be joined, three and three, beginning from A; that is, let arcs equal to A D be taken one after the other, A D representing three-tenths of the circumference, and let the chords of these arcs be drawn.

As the two numbers 3 and 10 are prime to each other, their least common multiple is equal to 3×10, or 30; consequently the sum of ten arcs equal to A D is equal to three times the length of the circumference, and it is the least multiple of the arc A D, which contains the circumference an exact number of times.

The chords of these ten arcs form therefore a closed polygonal line, because it begins at point A and terminates at that point.

This line has ten sides, and therefore ten summits, which are nothing else than the ten points of division of the circumference.

Therefore it forms a regular concave decagon, for its sides are evidently equal, as well as its angles, and each of its sides traverses its surface.

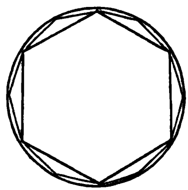
In the second place, let n be equal to 4, a number which is not prime with 10; that is, let the ten points of division of the circumference be joined, four and four.

I then return to the starting point A, after having

drawn the chords of five consecutive arcs, equal to arc A E, which represents the four tenths of the circumference. For the sum of the five arcs is equal to five times the four tenths of the circumference, or twice the length of this curve.

The regular concave polygon, formed in this manner, has therefore only five sides; that is, as many unities as there are in the quotient of the number 10 by the greatest common measure of the numbers 10 and 4.

The name of starlike polygon has been given to every regular concave polygon, on account of its form. There is only one starlike pentagon; there are three starlike heptagons, a single starlike octagon, a single starlike decagon, etc.



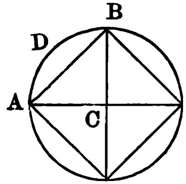
Scholium. — If a regular convex polygon is inscribed in a given circle—for example, a hexagon—and afterwards the regular polygons, whose number of sides is twice as great in regular increasing progression (that is, polygons of 12, 24, 48, etc., sides)—the perimeters of these polygons go on increasing, though still remaining less than the circumference in which they are inscribed, and to which they indefinitely approach.

The same thing is the case with their surfaces, which differ continually less from that of the circle.

These facts are expressed by saying that the circumference and the circle are the limits towards which tend the perimeter and the surface of a regular inscribed polygon, the number of whose sides increases indefinitely.

THEOREM III.

The ratio of the side of a square inscribed in a circle to the radius is $\sqrt{2}$.



Let A B be the side of the inscribed square. Draw the radii C A and C B. Arc A D B being a quadrant, the angle A C B is a right angle. In the right-angled triangle A B C we have, by the theorem of Pythagoras,* $AB^2 = AC^2 + CB^2$, and calling the radius r, $AB^2 = r^2 + r^2$, or $AB^2 = 2r^2$; whence $\frac{AB^2}{r^2} = 2$, and consequently $\frac{AB}{r} = \sqrt{2}$.

Scholium I.—In virtue of this theorem, the side of a square inscribed in a circle can be found if the radius is known, and the converse. For, from the equality $\frac{AB}{r} = \sqrt{2}$ results $AB = r\sqrt{2}$; consequently, $r = \frac{AB}{\sqrt{2}}$, or multiplying the two terms of this fraction by its denominator, in order that it may become rational, $r = \frac{AB\sqrt{2}}{2}$.

Scholium II.—The side of the square in a circle and its radius are incommensurable; therefore their ratio is incommensurable.

As the side of a square inscribed in a circle is the diagonal of a square whose side is the radius, it results that the *diagonal of a square and its side are incommensurable straight lines.*

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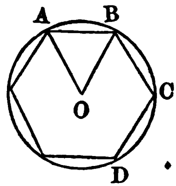
* Theorem XX., Book III.

THEOREM IV.

The side of the regular hexagon inscribed in a circle is equal to the radius of the said circle.

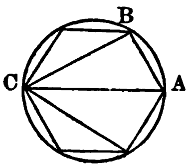


Let A B be a side of the regular inscribed hexagon. Draw the radii OA and OB; then AOB $= \frac{4R}{6} = \frac{2R}{3} = \frac{2}{3}R$; then the sum of A + B of the other angles of the triangle A B O will be worth $\frac{4}{3}R$; and as the two angles A and B are equal, being opposite to the equal sides A O and B O, each of these angles will be worth $\frac{2}{3}R$. Therefore, as angles A, O, and B are all equal, sides O B and A B and A O are equal.

Scholium.—It is proper to give also this enunciation of the above theorem: *The chord of the sixth part of the circumference, or of the arc of 60°, is equal to the radius.*

THEOREM V.

The ratio of the equilateral triangle inscribed in a circle to the radius is $\sqrt{3}$.



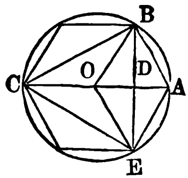
Let C B be the side of the inscribed equilateral triangle. Draw the diameter C A and the chord B A. As the arc C B is a third of the circumference, and arc C B A the half of the circumference, the arc A B will be equal C B A − C B = $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of the circumference, and therefore the chord A B is equal to the radius. In the right-angled triangle A B C we have, by the theorem of Pythagoras,

$CB^2 = AC^2 - AB^2$, and calling the radius r, $CB^2 = 4r^2 - r^2$, or $CB^2 = 3r^2$; whence $\frac{CB^2}{r^2} = 3$, and consequently $\frac{CB}{r} = \sqrt{3}$.

Scholium I.—According to this theorem we can find the side of the inscribed equilateral triangle, knowing the radius, and the converse; for, as $\frac{CB}{r} = \sqrt{3}$, it is inferred that $CB = r\sqrt{3}$, and that $r = \frac{CB}{\sqrt{3}}$, or multiplying the two terms of this fraction by its denominator, that it may be rational, $r = \frac{CB\sqrt{3}}{3}$.

Scholium II.—*The side of the equilateral triangle inscribed in a circle and the radius are incommensurable,* because their ratio is incommensurable.* (Theorem XVI. (*a*), Book II., Definition 46).

Scholium III.—The apotheme O D of the equilateral triangle inscribed in a circle is equal to half the radius; for the similar triangles B D O, C B A give B O : C A :: O D : A B; and as C O is the half of C A, O D also will be half A B or the radius.†



A straight line is said to be divided *into extreme and mean ratio* when it is divided into two parts, such that the greater part is a mean proportional between the said straight line and the lesser part.

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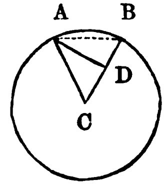
* Amiot calls it irrational. *Leçons Nouvelles,* "Geom. Plane," p. 274, Prob. 2, livre 4, ch. iv.

† A shorter reasoning is because O B A E is a lozenge or rhombus (Rouché and Comberousse, "Geom. Plane.")

THEOREM VI.

The side of a regular decagon inscribed in a circle is equal to the greater part of the radius, divided in mean and extreme ratio.



Let A B be a side of the regular inscribed decagon. Draw the radii C A and C B. Angle $C = \frac{4R}{10} = \frac{2}{5}R$; therefore, $A + B = \frac{8}{5}R$; and as the angles A and B are equal, from being opposed to equal sides, each of them will be worth $\frac{4}{5}R$.

This being established, divide the angle A into two equal parts by means of the straight line A D, and then :

$$BD : DC :: AB : AC \ (1).$$

But as angle C A D is equal to $\frac{2}{5}R$, the sides A D and C D are equal, because they subtend equal angles, C and C A D. Also the angle $ADB = C + CAD = \frac{4}{5}R$;* and therefore sides A B and A D, opposite to the equal angles A D B and A B D, are equal; therefore $DC = AB$; and proportion (1) will be : $BD : DC :: DC : BC$.

Therefore it is shown that the radius B C is divided into mean and extreme ratio at D, and that its greater segment D C is equal to the side A B of the regular inscribed decagon.

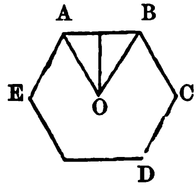
Scholium.—It has been seen (Theorem III., Book IV.) that the ratio of the side of a square inscribed in a circle to the radius is $\sqrt{2}$; therefore, if we take the radius as unity, the value of the side of the inscribed square is $\sqrt{2}$.

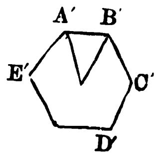
Consequently, if geometrical constructions were exact (which they are not), we should have exactly a straight line whose value would be $\sqrt{2}$. The same can be said of any other incommensurable square root; $\sqrt{n}$, which is the value of a straight line, being a mean proportional between the straight lines whose values are 1 and n.

* Theorem XXVIII., Corollary I., Book I.

THEOREM VII.

Regular polygons of the same number of sides are similar.





Let us consider, for instance, two regular hexagons, A B C, A′ B′ C′. As the six angles of a hexagon are equal to eight right angles,* each angle of the regular hexagon will be worth $\frac{8\,R}{6}$; therefore the angles of both polygons are equal.

As the sides of each polygon are equal, the ratio of one side of one polygon to one side of the other polygon is always the same; therefore the sides adjacent to equal angles of both polygons are proportional. Accordingly, the proposed polygons have their angles respectively equal, and their sides proportional; and therefore the said polygons are similar.

THEOREM VIII.

The perimeters of two regular polygons A B C D E, *a b c d e, of the same number of sides, are proportional to their radii* A O *and a o, and to their apothemes* O F *and o f.*†

Let P and p be the perimeters of two polygons; as

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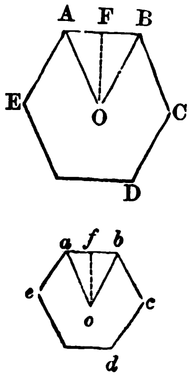
* Theorem XXIX., Book I.

† The names radius and apotheme of a polygon are given to the radius of the circumscribed and the radius of the inscribed circle. (See above, *Remark*, Theorem I., Book IV).

these polygons are similar (hyp.), we shall have :—

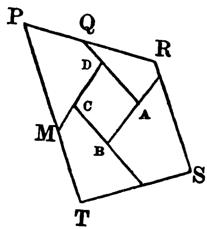
$$P : p :: AB : ab.$$



Now, if we draw the radii O B and ob, the triangles O A B and oab will be similar, because the angles O A B, oab are equal, as halves of the equal angles E A B, eab, and the angles O B A, oba will be equal for the same reason. Therefore A B : ab :: O A : oa :: O F : of; and consequently P : p :: A O : ao :: O F : of.

THEOREM IX.

Every convex closed line A B C D *enveloped by any other closed line* P Q R S T *is less than it.*



All the infinite lines A B C D, P Q R S T, etc., which enclose the plane surface A B C D, cannot be equal. For drawing the straight line M D, which does not cut A B C D, M D will be < M P Q D; and adding to both members the part M T S R Q D, the result will be M D Q R S T M < M P Q R S T M.

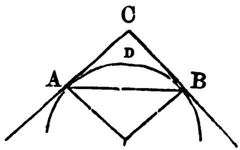
As all the lines, infinite in number, that enclose the plane surface A B C D are not equal, one of these two cases will result: either two or more of these lines will exist, which will be equal in magnitude, and less than all the others, or one only will exist, less than all the others.

It has been seen that M P Q R S T M is greater than M D Q R S T M; therefore P Q R S T M is not one of the smaller ones. The same thing can be demonstrated of any other of the lines mentioned, different from A B C D. Therefore, two of these cannot exist, being equal in magnitude, and less than all the others; and since no line different from A B C D is of these lesser lines, A B C D is the lesser line.

Corollary I.—*The circumference is greater than the perimeter of every inscribed polygon, and less than that of every circumscribed polygon.*

Corollary II.—*The perimeter of a regular inscribed polygon, which has twice as many sides as another regular inscribed polygon, is greater than the perimeter of the latter.*

Corollary III.—If from the point C outside the circle, two tangents be drawn to the same, terminating in the points of contact A and B, the sum of these two tangents is greater than that of the arc A D B included between them. For drawing the chord A B, we have by the theorem, $AB + AC + BC > ADB + AB$; therefore $AC + BC > ADB$.



THEOREM X.

Every curved line is the limit of a broken line, whose summits are on the curve, and whose sides can be multiplied till they come to be less than any given quantity.

For, as the sides of the broken line may come to be less than any given quantity, it is evident that the curve and the broken line become at length confounded in one; and

therefore the difference of their length can come to be as small as is desired; that is, the curve is the limit of the aforesaid broken line.

COROLLARY.—The circumference is the limit of the perimeters of the regular inscribed polygons; for if we inscribe a regular polygon, then another regular polygon of twice the number of sides, next another of twice as many sides as the last, and continue this operation indefinitely, the circumference will be divided into arcs less than any given quantity, and *à fortiori* the sides of the polygon, which are less than the said arcs, will come to be less than any assignable quantity; therefore the circumference is the limit of the said perimeters.

THEOREM XI.

The circumferences c, c' are proportional to their radii, r and r'.

Let p and p' be the perimeters of two similar regular polygons inscribed in the circumferences c and c'; we shall have, according to Theorem VIII., the proportion $\frac{p}{r} = \frac{p'}{r'}$. Now, as the circumferences c and c' are the limits of the perimeters p and p',* and consequently the constant quantities $\frac{c}{r}$ and $\frac{c'}{r'}$ are the limits of the variables $\frac{p}{r}$ and $\frac{p'}{r'}$, we shall have in virtue of the theorem of the limits: $\frac{c}{r} = \frac{c'}{r'}$

COROLLARY I.—*Circumferences are proportional to their diameters*—that is, $c : 2r :: c' : 2r'$; and therefore the ratio of one circumference to its diameter is equal to the ratio of any other circumference to its diameter. Therefore *the ratio of the circumference to the diameter is always the same, or is constant.*

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* Theorem X., Corollary.

This constant ratio, which is commonly designated by the Greek letter π, is equal to $3{\cdot}14159265358979\ldots$; the geometer Lambert proved in 1761 that this number is irrational, but his demonstration is not adapted for an elementary geometry.

Two thousand years before our time, Archimedes first discovered that π is included between the numbers $3\frac{10}{70}$ and $3\frac{10}{71}$. The larger quantity $3\frac{1}{7}$ is generally used, exceeding π by only a half-hundredth. Adrian Metius, a Dutch geometer of the 16th century, has given us a proximate value of the same ratio—the number $\frac{355}{113}$, which only differs from it by a half-millionth, and which is remarkable for the manner in which it is formed by the three first uneven numbers, 1, 3, and 5.

It results from the fact that the ratio of a circumference to its diameter is constant; that to compute *the length of a circumference of which the diameter is given, you must multiply the diameter by the number π; and reciprocally, to calculate the magnitude of the diameter* of a given circumference, you must divide this circumference by the number π.

These two rules are comprised in the following formula: Circle $R = 2R \times \pi$; that is, circle $R = 2\pi R$.

COROLLARY II.—*To calculate the length l of an arc of n degrees, the radius R of this arc being given.*

The circumference described with the radius R being equal to $2\pi R$, the arc of a degree, which is the three hundred and sixtieth part of it, has for its measure $\frac{2\pi R}{360}$ or $\frac{\pi R}{180}$; the arc of n degrees has therefore for its measure n times $\frac{\pi R}{180}$, or $\frac{\pi R n}{180}$.

Therefore we have the formula: $l = \frac{\pi R n}{180}$, which serves to calculate one of the three quantities l, R, n, when the two others are given.

gons be constructed, the polygon constructed on one side will be equivalent to the difference of the polygons constructed on the hypothenuse and the other side.

This is demonstrated in the same manner as the previous theorem.

APPENDIX TO BOOK III.

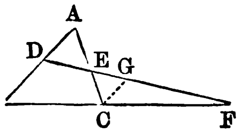
Theory of Transversals.

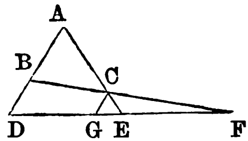
DEFINITION.

67. The name of transversal is given to a straight line which meets a system of several lines.

THEOREM I.

When the three sides of a triangle, produced if necessary, are intersected by a transversal, there are on each side two segments, and these six segments are such that the product of three of them, not having any extremity in common, is equal to the product of the others.





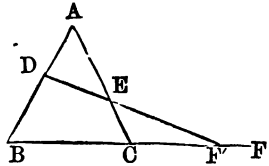
Let A B C be the proposed triangle, and D E F the transversal. Draw through point C a parallel C G to the side A B, till it meets the transversal. The similar triangles A D E, E G C give the proportion: $\frac{AE}{CE} = \frac{AD}{CG}$; whence $AE \times CG = CE \times AD$ (1).

The similar triangles C G E, F B D give also the pro-

portion $\frac{CF}{BF} = \frac{CG}{BD}$, from which is deduced $CF \times BD = BF \times CG$ (2). Multiplying the equalities (1) and (2), and dividing by the common factor CG, we have: $AE \times CF \times BD = AD \times BF \times CE$.

THEOREM II.

Reciprocally, if three points taken in equal number on the sides of a triangle and in unequal number on its sides produced, determine six segments, such that the product of three of them that are not consecutive be equal to the product of the three others, these three points will be in a straight line.



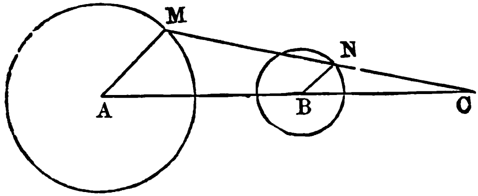
Let ABC be the triangle, and let D, E, F be three points, such that $AD \times BF \times CE = BD \times CF \times AE$ (1); then the three points D, E, F will be in one straight line; for if the straight line DE met BC on a point F' different from F, we should have, by the preceding theorem, $AD \times BF' \times CE = BD \times CF' \times AE$ (2). Dividing the equality (1) by the equality (2), and suppressing the common factors, we have: $\frac{BF}{BF'} = \frac{CF}{CF'}$; whence $\frac{BF - CF}{BC - CF'} = \frac{BF}{BF'}$; whence again, $\frac{BC}{BC} = \frac{BF}{BF'}$; an evidently false proportion, unless point F is confounded with point F'.

Remark.—This theorem often gives the means of detecting, very simply, if three points are in a straight line; we are about to give an example of this.

LEMMA I.

Two circumferences A *and* B *being given, if two radii* A M, B N *be drawn, parallel and in the same direction, the straight line* M N, *which joins their extremities, will meet the line of the centres at a point* C *that will be the same, whatever way the direction of the parallel radii* A M, B N.

For the similar triangles A M C, B N C give the proportion $\frac{AC}{BC} = \frac{AM}{BN}$; whence $\frac{AC - BC}{BC} = \frac{AM - BN}{BN}$, or $\frac{AB}{BC} = \frac{AM - BN}{BN}$.



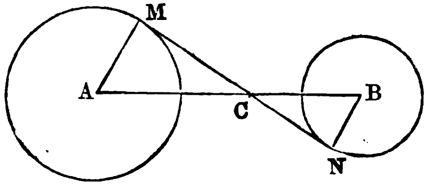
But A B, A M − A N, B N are constant quantities; therefore B C will also have a constant value.

This point C is the *direct centre of similitude* of the two circumferences, and its position on the line of the centres is determined by the proportion $\frac{AC}{BC} = \frac{AM}{BN}$. It would be easily seen also that this is the meeting-point of all the common exterior tangents.

LEMMA II.

Given two circumferences A *and* B, *if the radii* A M, B N *are drawn, parallel and in an opposite direction, the straight line* M N *will meet the line of the centres at a point* C, *which will be the same, whatever may be the direction of the radii* A M, B N.

For the similar triangles A M C, B N C give the proportion $\frac{AC}{CB} = \frac{AM}{BN}$.

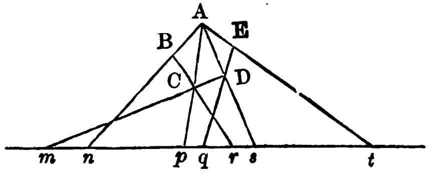


The point C divides the straight line A B in the ratio of the two radii; therefore this point is constant. *This point is the inverse centre of similitude* of the two circumferences. It may also be seen that it is the meeting-place of the common interior tangents.

THEOREM III.

If the sides of a plane polygon, or the sides produced, are cut by a transversal, there will be on each side two segments formed by the transversal, such that the product of the segments which have no common extremity will be equal to the product of all the others.

Let the pentagon A B C D E be cut by a transversal $mnpqr$; then, $An \cdot Br \cdot Cm \cdot Dq \cdot Et = Bn \cdot Cr \cdot Dm \cdot Eq \cdot At$.



To prove this, from one of the summits A of the polygon let diagonals be drawn to all the other summits, and let them be produced till they meet the transversal.

The triangles A B C, A C D, A E D, considered separately, as cut by the transversal, will give :—

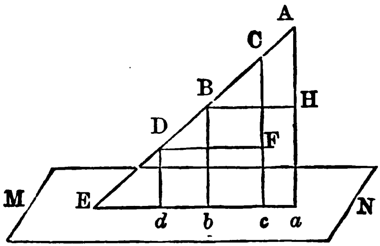
$$\mathrm{A}n \,.\, \mathrm{B}r \,.\, \mathrm{C}p = \mathrm{B}n \,.\, \mathrm{C}r \,.\, \mathrm{A}p.$$
$$\mathrm{A}p \,.\, \mathrm{C}m \,.\, \mathrm{D}s = \mathrm{C}p \,.\, \mathrm{D}m \,.\, \mathrm{A}s.$$
$$\mathrm{A}s \,.\, \mathrm{D}q \,.\, \mathrm{E}t = \mathrm{D}s \,.\, \mathrm{E}q \,.\, \mathrm{A}t.$$

Multiplying these equalities and cancelling the common factors, we obtain : $\mathrm{A}n \,.\, \mathrm{B}r \,.\, \mathrm{C}m \,.\, \mathrm{D}q \,.\, \mathrm{E}t = \mathrm{B}n \,.\, \mathrm{C}r \,.\, \mathrm{D}m \,.\, \mathrm{E}\ t \,.\, \mathrm{A}\ t.$

THEOREM IV.

If the different portions of a straight line be projected on a plane, each projection will be equal to the portion of the straight line corresponding to it, multiplied by a constant number for all these straight lines.

Let the portions AB, C D of the straight line A E be projected on the plane M N, and let ab, cd be their projections; then let D F and B H be drawn parallel to E a.



The similar triangles C D F, A B H give the proportion $\frac{\mathrm{DF}}{\mathrm{DC}} = \frac{\mathrm{BH}}{\mathrm{AB}}$; or because dc = D F and ba = B H, $\frac{dc}{\mathrm{DC}} = \frac{ab}{\mathrm{AB}}$; therefore, if $\frac{dc}{\mathrm{DC}} = m$, whence $dc = \mathrm{DC} \times m$, we shall have $\frac{ab}{\mathrm{AB}} = m$, whence $ab = \mathrm{AB} \times m$.

THEOREM V.

If the sides of a polygon turned to the left, or their sides produced, are cut by a transversal, there will be on each side two segments, such that the product of all those segments that have not a common extremity will be equal to the product of all the others.

Let $\alpha\beta\zeta\gamma\delta$ be the proposed polygon, and let ϵ, λ, μ, ρ be the points where the sides $\alpha\beta$, $\beta\gamma$, $\delta\gamma$, $\alpha\delta$ meet the transversal plane. Let this polygon be projected on a plane perpendicular to the transversal plane.

Let it be supposed that A B C D is this projection, and let S N be the intersection of the plane of projection and of the transversal plane. If the sides of the polygon A B C D be produced till they meet S N, this gives, by Theorem IV., A E . B L . C M . D R = B E . C L . D M . A R (1).

But A E and B E are the projections on one and the same plane of the segments $\alpha\epsilon$, $\beta\epsilon$ of the polygon turned left; this gives—

$$\begin{aligned} AE &= \alpha\epsilon \,.\, m, & BE &= \beta\epsilon \,.\, m; \\ \text{also } BL &= \beta\lambda \,.\, n, & CL &= \gamma\lambda \,.\, n. \\ CM &= \gamma\mu \,.\, p, & DM &= \delta\mu \,.\, p. \\ DR &= \delta\rho \,.\, r, & AR &= \alpha\rho \,.\, r. \end{aligned}$$

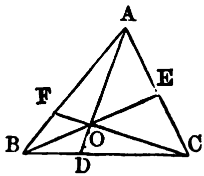
Substituting in equality (1) for the segments A E, B E, B L, C L . . . their equivalents, and dividing on both sides by $m \,.\, n \,.\, p \,.\, r$., we obtain $\alpha\epsilon \,.\, \beta\lambda \,.\, \gamma\mu \,.\, \delta\rho = \beta\epsilon \,.\, \gamma\lambda \,.\, \delta\mu \,.\, \alpha\rho$. Q. E. D.

THEOREM VI.

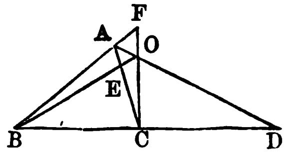
If a point O *taken in the plane of a triangle* A B C *be joined to the three summits of this triangle, these straight lines determine on the sides of the triangle, or on the sides produced, six segments, such that the product* A F . B D . C D *of three segments, non-consecutive, is equal to the product* F B . D C . A E *of the three others.*

For the triangle A B D cut by the transversal F C gives the equality A F . B C . D O = B F . D C . A O (1).

In like manner, the triangle A D C cut by the transversal B E gives A O . B D . E C = O D . B C . A E (2).



Multiplying member by member the equalities (1) and (2), and cancelling the common factors, we have A F . B D . E C = B F . D C . A E.



(The further development of the Theory of Transversals is reserved for a special Treatise on Modern Geometry, with a popular view of the recent improvements introduced by M. Chasles.)

EXERCISES TO THE FIVE BOOKS.

EXERCISES ON BOOK I.

1. The bisectors of two adjacent and supplementary angles are perpendicular one to the other. (The bisector of an angle is the line which divides this angle into two equal parts.)

2. When four adjacent angles are equal together to four right angles, if the first is equal to the third and the second equal to the fourth, the sides of these angles are, two and two, in the same straight lines.

3. The bisectors of two angles opposed at the summit are in a straight line.

4. The sum of the straight lines that unite a point taken in the interior of a triangle to the three summits, is less than the perimeter of the triangle, and greater than the half of that perimeter.

5. The straight line which joins the summit of a triangle to the middle of the opposite side is less than the half of the sum of the two other sides, but greater than the half of the excess of that sum over the third side.

6. Conclude from this theorem that the sum of the straight lines which join the summits of a triangle to the centres of the opposite sides is less than the perimeter of the triangle, and greater than the half of the perimeter.

7. If you prolong the sides B A, C A of the triangle B A C beyond the summit A, by quantities A B′, A C′, which are respectively equal, and if the straight line B′C′ be drawn, first, the centres of the lines C B, B′ C′ and the

summit A will be in one straight line ; secondly, the last of these three points will divide the distance of the two others into equal parts.

8. If there be taken on one of the sides of any angle A B C the lengths B D, B E, and on the other side the lengths B D′, B E′, respectively equal to B D, B E, and if afterwards the straight lines D E′, E D′ be drawn, these lines will intersect on the bisector of the angle A B C. From this results the means of constructing the bisector of an angle.

9. The perpendiculars drawn from the summits of an equilateral triangle to the opposite sides are equal.

10. If two points equally distant from the summit are taken on the equal sides of an isosceles triangle, and they are joined by straight lines to the opposite extremities of the base, these lines will intersect on the straight line, which passes from the summit to the middle of the base.

11. The straight lines which join the summits of an isosceles triangle to the centres of the opposite sides intersect at the same point.

12. The perpendiculars drawn from the extremities of the base of an isosceles triangle on to the opposite sides are equal.

13. Find on a straight line a point, such that the sum or the difference of its distances from two given points is a *minimum* or a *maximum*. Prove that the straight lines on which these distances are measured are equally inclined towards the given straight line.

14. The perpendiculars raised on the sides of a triangle, through the centres of its sides, intersect in the same point.

15. The bisector of an angle is the geometrical locus of the points equally distant from the sides of this angle.

16. The bisectors of the angles of a triangle meet at the same point.

17. If the bisector of the angle of a triangle divides the opposite side into two equal parts, this triangle is isosceles.

18. If through the point of intersection of the bisectors of the angles of a triangle, a parallel be drawn to one of its sides, this straight line will be equal to the sum of the

segments intercepted on the two other sides by the two parallels.

19. If through the summits of a triangle, parallels are drawn to the opposite sides, these straight lines will determine a second triangle equal to four times the first. What is the relation of the parallel sides?

20. The perpendiculars dropped from the summits of a triangle on the opposite sides meet at the same point.

21. The bisectors of the angles of a quadrilateral form another quadrilateral, of which the opposite angles are supplementary.

22. If the opposite sides of a quadrilateral be produced till they meet, the bisectors of the two angles formed by them intersect at an angle equal to half the sum of two opposite angles of the quadrilateral. In which case are these bisectors perpendiculars?

23. The bisectors of two angles which have their sides parallel are parallel or perpendicular. The same is the case with the bisectors of two angles whose sides are perpendicular.

24. In every convex quadrilateral, first, the bisectors of two consecutive angles intersect under an angle equal to half the sum of the two others; secondly, the bisectors of two opposite angles form an angle equal to half the difference of the two other angles of the quadrilateral.

25. The parallelogram formed by drawing, through the extremities of each diagonal of a quadrilateral, parallels to the other diagonal, is equivalent to twice that quadrilateral. Deduce from this theorem that two quadrilaterals are equivalent, if their diagonals are equal, each to each, and equally inclined one to the other.

26. Every straight line which passes through the point of intersection of the diagonals of a parallelogram is divided by this point into two equal parts, and this line divides in its turn the parallelogram into two equal parts. For this reason, the diagonals at the point of intersection of the diagonals receives the name of centre of the parallelogram.

27. The diagonals of two parallelograms *inscribed one within the other*, that is, such that the summits of the one are on the sides of the other, pass through the same point.

28. The sum of the perpendiculars drawn from any point of the base of an isosceles triangle to the two other sides is constant. The difference of the perpendiculars drawn from any point of the prolongations of the base to the two other sides is constant. How must the enunciation be altered in the case of a point exterior to the triangle ?

29. Prove, first, that in a rectangle, parallelograms may be inscribed of which the sides are respectively parallel to the diagonals of the rectangle ; secondly, that the perimeter of these parallelograms is equal to the sum of the diagonals of the rectangle.

30. Given a rectangle and a point situated in the interior of a quadrilateral: if the given point is supposed to be a ball, infinitely small, and the perimeter of a rectangle as a material line, perfectly elastic, so that when the ball strikes it, it always recoils, making the angle of incidence equal to the angle of reflexion ; find the direction in which the ball must be projected to return to the starting-point, after having touched the four sides of the rectangle. What is the length of the distance traversed by the ball?

EXERCISES ON BOOK II.

31. The greatest and the least of all the straight lines that can be drawn from a point to a circumference pass through the centre.

32. A straight line and a point being given, describe, with a given radius, a circumference of which the centre is situated on the straight line, so that the sum of the maximum and minimum distances from the point to this circumference is equal to a given length.

33. If two arcs A B, C D of the same circumference are equal, their chords A B, C D, and the straight lines A C, B D which join cross-shape the extremities of these arcs, intersect on the same diameters.

34. Describe, with a given radius, a circumference passing through two given points.

35. Describe, through a given point, a circumference passing at the same distance from three given points, not in a straight line.

36. Describe, with a given radius, a circumference passing at the same distance from three given points, not in a straight line.

37. Given on a map four points, of which three are not in a straight line, draw on this map a circular path passing at an equal distance from each of these points.

38. Describe, with a given radius, a circumference intercepting, on two straight lines, chords of which the length is given.

39. A straight line, movable on a plane, can be brought from any one of its positions to another by a rotation round a point of the plane, provided its two positions are not parallel and in the same direction.

40. A triangle and any plane figure, in general movable in a plane, can be brought from any of their positions to another by a rotation round a point in the plane, provided that in these two positions the equal sides be not parallel and in the same direction.

41. If the chord of an arc be divided into three equal parts, the radii which pass through the points of division do not divide the arc in three equal parts.

42. Which is the geometrical locus of the centres of the chords of a circumference equal to a given straight line?

43. Draw, through a given point, a circumference touching a straight line at a given point.

44. Describe, through two given points, a circumference touching a parallel to a straight line drawn through the given points.

45. Describe a circumference which intercepts chords of a given length on two parallel straight lines.

46. The straight lines which join the extremities of two parallel chords intersect on the diameter perpendicular to these chords.

47. Let A be the centre of a circle. If the radius A B be prolonged by a quantity B C equal to A B; then, if from a point C the perpendicular C D be dropped on any tangent to the circle; and, next, if the straight line be drawn joining the foot D of this perpendicular to the

extremity B of the radius A B, the angle A B D exterior to the triangle B C D is constantly equal to three times the interior angle B D C.

48. When two circumferences have no point in common, the least and the greatest of the straight lines that can be drawn from one circumference to the other pass through the centres of those circumferences.

49. If through one of the points of intersection of two circumferences, a parallel be drawn to that straight line which joins their centres, the sum of the chords intercepted on this parallel is equal to twice the distance of the centres. When the secant is caused to turn round the point of intersection of the circumferenees, how does the sum of the intercepted chords vary?

50. What is the geometrical locus of the centres of the circumferences which, described with the same radius, divide a given circumference into two equal parts?

51. Describe a circumference which, passing by a given point, touches a given circumference at a given point.

52. Describe a circumference which passes by two given points and cuts a given circumference, so that the common chord is parallel to a given chord.

53. Describe, with a given radius, a circumference which passes by a given point, and of which the shortest distance to a given circumference is a known length.

54. From the summits of a triangle as centres describe three circumferences, such that each of them touches the two others.

55. What is the geometrical locus of the summit of an angle which moves in a plane so that its sides pass through two given points?

56. If the opposite angles of a convex quadrilateral are supplementary, the four summits are situated on the same circumference; that is, the quadrilateral is *inscriptible.*

57. If a convex polygon, inscribed in a circle, has an even number of sides, the sum of its angles of the even order is equal to the sum of its angles of uneven order. The reciprocal proposition is only true in the case of the quadrilateral.

58. What is the geometrical locus of the centres of

the chords intercepted by the circumference on all the secants that can be drawn through a given point?

59. If, from a point of the circumscribed circumference, perpendiculars be dropped to the sides of a triangle, the feet of these perpendiculars are in one straight line.

60. The feet of perpendiculars drawn from the summits of a triangle on the opposite sides are the summits of a second triangle, whereof the angles have for bisectors the heights of the other triangle.

61. If four circumferences are drawn, such that each of them passes by two consecutive summits of an inscribed quadrilateral, these curves intersect at four points different from those of the quadrilateral. Prove that these four points belong to one and the same circumference.

62. When the sides of an angle cut two circumferences, the chords of the arcs which they intercept on one of these curves, being indefinitely produced, make a quadrilateral inscriptible with the chords of the arcs intercepted on the other curve.

63. If a secant be drawn through one of the points of intersection of two circumferences, it cuts these circumferences at two other points, of which the tangents make a constant angle.

64. If three points A, B, C divide a circumference into three equal parts, the distance from every point M of the arc A B to the point C is equal to the sum of the distances from the same point M to the two points A and B.

65. Construct a triangle of which an angle is known; also one of the adjacent sides, and the length of the straight line joining the centre of this side to the opposite summit.

66. Construct a triangle in which the two sides are given; also the length of the straight line which joins the middle of one of them to the opposite summit.

67. Construct a triangle of which one side is known, one of the adjacent angles, and the length of the bisector.

68. Construct a triangle with a given angle, one of the adjacent sides, and the sum or difference of the two other sides.

69. Construct a triangle in which an angle is given; also the side opposite to it, and the sum or difference of the two other sides.

70. Construct a parallelogram of which one angle and the two adjacent sides are known.

71. Construct a rectangle of which two consecutive sides are known.

72. Given the diagonals of a lozenge, construct that quadrilateral.

73. Construct a trapezium of which the sides are given. (A trapezium is any quadrilateral with two parallel sides.)

74. Given two parallel straight lines and a point, draw through the point a secant, such that the portion of that line comprised between the two parallels is of a given length.

75. Construct a triangle in which the base, the height, and the straight line which joins the middle of the base to the opposite summit are given.

76. Draw between two circumferences a straight line having a given length, and parallel also to a given direction.

77. Describe, from a given point as centre, a circumference which intercepts on another circumference an arc having a chord of a given length.

78. Construct a triangle of which two angles and one of the three heights are known.

79. Construct a trapezium of which the diagonals and the parallel sides are given.

80. Construct a triangle of which two sides and one of the heights are known.

81. Inscribe in a circle an angle of a given magnitude, so that one of its sides passes by a given point, and that the other may be parallel tó a given straight line.

82. Describe, with a given radius, a circumference tangent to two given straight lines, or to a given straight line and circumference.

83. The diagonals of a parallelogram and their angle being given, construct this quadrilateral.

84. Construct a triangle of which two angles and the sum of the two sides are given.

85. Construct a triangle of which two angles and the perimeter are known.

86. If a circle be inscribed in an angle, the two points of contact divide the circumference into two arcs, such that the whole tangent to the smaller of those arcs forms a triangle with a constant perimeter, with the sides of the angle.

87. Through a point M, given in an angle B A C, draw a secant, such that the perimeter of a triangle formed by this line and the sides of the angle may have a given length.

88. Draw a secant common to two circumferences, in such sort that the intercepted chords may have given lengths.

89. Two concentric circumferences being given, draw a triangle of which two angles are given, and which has two summits on one of the circumferences, and the third on the other.

90. Construct a triangle in which one of the three heights, one angle, and its bisector are known.

91. Construct a triangle of which the perimeter, an angle, and the corresponding height are given.

EXERCISES ON BOOK III.

92. The straight line which joins the centres of two sides of a triangle is parallel to the third side, and equal to half that side.

93. The straight lines which join the centres of the consecutive sides of a quadrilateral form a parallelogram. In what cases is this parallelogram a rectangle, a lozenge, or a square?

94. Calculate within 0·001 mètres, each of the segments determined by the bisectors of the angles of a triangle on its sides, equal respectively to 12, 15, and 18 mètres.

95. Given any triangle A B C, side A C is diminished by an arbitrary quantity A A′, and side B C is increased by a quantity equal to B B′. Prove that the new base A′ B′ is cut by the old base A B in a ratio inverse to the primitive sides A C, B C.

96. Let there be two similar triangles A B C, A′ B′ C′, of which the homologous sides B C, B′ C′ are parallel. If the triangle A B′ C′ turns in its plane around their common summit A, what is the locus described by the point of intersection of the two straight lines which join the homologous extremities of the sides B C, B′ C′?

97. Given in the same plane two similar polygons, prove that there exists in this plane a point, such that the straight lines drawn from this point to any two homologous summits make a constant angle; and construct this point.

98. Inscribe a square in a semicircle, or in a triangle.

99. Inscribe in a circle an isosceles triangle, of which the sum or the difference of the base and height are given.

100. The straight lines drawn from the summits of a triangle to the centres of the opposite sides meet in the same point, which divides each of those lines in the ratios 2 to 1, starting from each summit.

101. If three straight lines pass through the same point, the ratio of the distances from any point of one to the other two is constant. Deduce from this theorem the geometrical locus of a point of which the distances to two given straight lines are proportional to lengths also given.

NUMERICAL PROBLEMS.

102. Two circles, of which the radii are respectively 0·5 mètres and 1·5 mètres in length, intersect in such wise that the tangents drawn through one of the points of intersection are perpendicular. Demanded the distance of their centres.

103. The radii of two concentric circles are 36 mètres and 20 mètres in length; in the great circle a chord is drawn tangent to the little one; required the distance of their centres.

104. The sides of the right angle of a right-angled triangle are equal to 16 mètres and 24 mètres. Required the projections of the sides of the right angle on the hypothenuse, and the distance from the summit of this angle to the opposite side.

105. Calculate the heights, the medial lines, and the bisectors of a triangle of which the sides are equal to 16 mètres, 25 mètres, and 39 mètres.

106. Calculate the length of the chord common to two circles of which the radii are 12 mètres and 15 mètres in length, knowing that the distance of their centres is 18 mètres.

GRAPHIC PROBLEMS.

107. If, from a point taken in the plane of a circle, two secants are drawn perpendicular one to the other, the sum of the squares of the distances from this point to the four points of intersection of the circumference and of the secants is constant.

108. The geometrical locus of the point, such that the sum of the squares of its distances from two fixed points is constant, is a circumference of which the centre coincides with the middle of the straight line which joins the two fixed points.

109. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the medial line of this last side on its direction.

The geometrical locus of the point, such that the difference of the squares of its distances from two fixed points is constant, is a straight line perpendicular to that which joins the fixed points.

110. Draw through two given points a circumference dividing into two equal parts a given circumference.

111. Describe a circumference passing by two given points, and touching a given straight line.

112. Describe, through a given point, a circumference touching two given straight lines.

113. The geometrical locus of a point, such that the tangents drawn from that point to two given circles are equal, is a perpendicular to the straight line joining the centres. This locus is known by the name of *radical axis of the two circles.*

114. The radical axis of three circles considered, two and two, meet at the same point, which is called the *radical centre of the three circles.*

115. Draw, through two given points, a circle touching a given straight line or circle.

116. What is the geometrical locus of the centres of the circles which intersect *orthogonally*—that is, forming a right angle—two given circles? All the circles which cut orthogonally two given circles have the same radical axis.

117. Find on the straight line which joins the centres of two circles, two points, such that the product of their distances from the centre of each circle is equal to the square of the radius of this circle.

118. Describe a circumference which cuts orthogonally three given circumferences.

119. A point A, a straight line, and a circle being given, find on the straight line a point B, such that the tangent to the circle, drawn through this point, is equal to the distance B A.

120. If, from a point of the radical axis of two circles, tangents are drawn to two other circles concentric with the first, the difference of the square of these tangents is constant.

121. The circles described on the diagonals of a trapezium as diameters have a common chord, which passes through the intersection of the non-parallel sides of the trapezium.

122. If a right angle turns round its summit, supposed to be fixed, what is the geometrical locus of the centres of the chords of the arcs which it intercepts on a circumference situated in its plane?

EXERCISES ON BOOK IV.

GENERAL PROBLEMS.

123. From a point taken in the plane of an angle, draw a straight line that may be divided in a given ratio through this point and the sides of the angle prolonged beyond the summit, if necessary.

124. From a point taken in the plane of an angle, draw a straight line divided by this point and the sides

of the angle into two segments, of which the product is equal to the square of a given line.

125. Inscribe in a circle a triangle, such that its sides, prolonged if necessary, may pass through two given points A, B, and intercepting on the circumference an arc of which the chord is parallel to the straight line A B.

126. From the extremity A of the diameter A B of a circle draw a secant, such that the sum or the difference of the distances from point A to the two points at which this secant cuts the circle, and the tangent drawn to the other extremity B of the diameter A B, is equal to a given line.

127. Construct a polygon that is similar to a given polygon, and of which the perimeter has a given length.

128. Construct a parallelogram that shall be similar to a given parallelogram, and of which the sides cut a given straight line at four given points.

129. Draw, through two given points, two parallels forming, with two given parallels, a parallelogram of which the sides are proportional to two lines *m* and *n*.

130. Draw on the plane of a triangle a straight line, such that the distances from the summits of this triangle to this straight line may be proportional to given lines *m*, *n*, *p*.

131. Draw two circles that shall be tangent one to the other, and touch a straight line at two given points, the sum or difference of their radii being equal to a given line.

NUMERICAL PROBLEMS.

132. Calculate, within a millimètre, and without using logarithms, the circumference which has for radius the diagonal of a square, whose side is 0·5 mètres, and show that the approximation required has been obtained.

133. Calculate, within a kilomètre, the radius of the circumference of the earth.

134. Calculate the radius of an arc of 20° 15′, of which the length is 8·50 mètres.

135. Two arcs of the same length have been described with radii of 0·25 and 0·18 mètres. One is an arc of 15° 20′; what is the number of degrees of the other?

GRAPHIC PROBLEMS.

136. Describe a circumference equal to the sum or difference of two given circumferences.

137. If two similar and regular polygons are placed so that one is inscribed and the other circumscribed round the same circle, the circumference of this circle is a proportional mean between the circumference inscribed in the first polygon and the circumference circumscribed round the second.

138. If a circle is made to revolve in a second circle of twice its radius, so that they are always tangents, a point of the circumference of the movable circle will describe a diameter of the fixed circle.

139. When a circumference is divided into equal parts by the points A, B, C, etc., and when, starting from A, those points are joined 2 and 2, 3 and 3, and in general h and h by straight lines, a regular concave polygon is formed, having n sides, if the numbers n and h are prime to each other; but if these numbers are not prime to each other, and d is their greatest common measure, the regular concave polygon has only $\frac{n}{d}$ sides.

140. There are as many regular polygons of n sides as there are unities in the half of the number which expresses how many whole numbers there are less than n, and prime to it.

141. The sum of the interior angles, formed by the consecutive sides of a regular polygon of n sides, is equal to as many times two right angles as there are unities in $n - 2h$, h being the number of times that the arc subtended by the side of the polygon contains the nth part of the circumscribed circumference.

The sum of the external angles, formed by each side, and the prolongation of the preceding side, is equal to $4h$ right angles.

142. Calculate the side and the apotheme of the regular convex octagon in the function of its radius. Apply the two formulæ, supposing the radius = 4·50 mètres.

143. Calculate the side and the apotheme of the

regular convex dodecagon in the function of its radius. Apply the two formulæ, supposing the radius = 1·50 mètres.

144. Prove that the ratio of a circumference to its diameter is comprised between the numbers 3 and 4, by the mere consideration of the perimeters of the regular inscribed hexagon, inscribed in this circumference, and of the circumscribed square.

145. Verify that the sum of the sides of the square and of the equilateral triangle, inscribed in the same circle, exceeds half the circumference of this circle by a quantity less than a half-hundredth part of the radius.

146. If a right-angled triangle be constructed, in which the sides of the right angle are equal to the diameter of a circumference, and to the excess of three times the radius over the third of the side of the inscribed equilateral triangle, then the hypothenuse of this right-angled triangle represents half of this circumference, within 0·0001 of the radius.

147. Given a circumference and a point, draw from this point a secant dividing the circumference into two arcs proportional to the numbers 11 and 15.

148. The side of an equilateral triangle circumscribed round a circle is twice the side of the equilateral triangle inscribed in this circle.

149. The apotheme of the regular hexagon inscribed in a circle is equal to half the side of the equilateral triangle inscribed in the same circle.

150. If the distance of the centres of two circles which intersect at right angles is equal to twice one of the radii, the common chord is the side of the regular hexagon inscribed in one of these circles, and the side of the equilateral triangle inscribed in the other.

151. What is the geometrical locus of the points, such that the sum of the squares of the distances of each of them from the summits of a regular polygon, which has an even number of sides, is constant?

152. Describe a circumference, such that the perimeter of the square inscribed in this curve is equal to that of the equilateral triangle circumscribed round a given circumference.

153. Construct a lozenge, of which the side has a given

length, and is also a mean proportional between the two diagonals.

154. Construct a square, of which the sum or difference of the diagonal and of the side are known.

EXERCISES ON BOOK V.

NUMERICAL PROBLEMS.

155. Calculate within a square centimètre the area of a rectangle of which the base is = 10·75 mètres and the diagonal = 15·25 mètres.

156. Calculate one of the heights and the area of the triangle of which the sides are respectively = 1·20 mètres, 1·85 mètres, and 2·25 mètres.

157. The area of a trapezium is equal to 2034·60 square mètres; its height is 18·40 mètres, and its inferior base 54·48 mètres. Calculate its upper base to within a centimètre.

158. Calculate in hectares (French measure) the area of a regular hexagon, of which the side is 450 mètres in length.

159. Calculate within a square centimètre the area of a regular octagon inscribed in a circle, of which the radius is 2·25 mètres.

160. Calculate in hectares the area of a lozenge, of which the side is equal to the smaller diagonal, knowing that the length of each of these lines is 20·50 mètres.

161. Calculate within a centimètre the side of the square equivalent to the equilateral triangle, of which the apotheme is 2·50 mètres.

GRAPHIC PROBLEMS.

162. The area of the trapezium is equal to the product of one of the non-parallel sides by the distance of this side from the middle of the opposite side.

163. Draw through the summit C of a triangle A B C a straight line M N, such that the trapezium

formed by it with the side A B and the perpendiculars drawn from the other summits A, B on M N is equivalent to a given square.

164. If the angles A, A′ of the two triangles A B C, A′ B′ C′ are equal or supplementary, the areas of these triangles are proportional to the products A B × A C, A′ B′ × A′ C′ of the sides formed by the angles A, A′.

165. Change a right-angled triangle into an equivalent isosceles triangle having an angle common with it. How many solutions are there to this problem?

166. Change a regular polygon into another regular polygon equivalent to it, and having twice as many sides.

167. Divide a triangle into two equivalent parts by a straight line perpendicular to one of its sides.

168. Divide a triangle into three parts proportional to given lengths, by joining a point in the interior to all the summits. Give the particular case in which the three given lines are equal.

169. Inscribe in a circle a trapezium of which the height and the surface are given. (The magnitude of a surface is determined by the side of the square equivalent to it.)

170. Given the position and the length of two straight lines, find the position of a point, such that, by joining it to the extremities of these lines, two triangles are formed of which the areas are proportional to two given straight lines M and N. Examine the case when M and N are equal.

171. Through a given point in the plane of an angle draw a secant, such that the product of the distances from the summit of the angle to the two points of intersection is equal to a given square.

172. Through a point given in the plane of an angle draw a secant, such that the area of the triangle made by it with the sides of that angle is equal to a given square.

173. The product of two sides of a triangle is equal to the product of the height perpendicular to the third side, by the diameter of the circumscribed circle.

174. Divide a triangle into two equivalent parts by a parallel to a given straight line.

175. Draw through one summit of a quadrilateral

a straight line which divides its surface into two equivalent parts.

176. If, in any quadrilateral, through the centre of each of the diagonals, a parallel be drawn to the other, and if their meeting-point be joined to the centres of the sides of the quadrilateral, it will be divided into four equivalent quadrilaterals.

177. Prove that the square constructed on the diagonal of a square is twice the proposed square.

178. The square on the sum of two straight lines is equivalent to the sum of the squares on each of the lines, increased by twice their rectangle.

179. The rectangle constructed on the sum and the difference of two straight lines is equivalent to the difference of the squares of those lines.

180. Make a square equal to the sum or the difference of two squares.

181. Given two similar polygons, construct a polygon similar to them, and equivalent to their sum or their difference.

182. Construct a triangle similar to a given triangle, and of which the summits are placed on three concentric circumferences, or on three parallel straight lines.

183. Divide a triangle in any number of equivalent parts, by parallels to one of its sides.

184. Construct an equilateral triangle equivalent to the sum or the difference of two given polygons.

185. Inscribe in a given triangle a triangle similar to another given triangle.

186. Draw through a given point a straight line which divides the surface of a trapezium into two parts proportional to given lines m and n.

187. Construct on a given base a triangle equivalent to a given polygon, and such that the straight line which joins its summit to the middle of the base is a mean proportional between the two other sides.

188. Given two parallel straight lines and two points, draw through these points two straight lines that intersect on one of the parallels, and form with the other a triangle equivalent to a given square.

189. Divide a trapezium in any number of equivalent parts by parallels to its bases.

190. Divide by a parallel to the base, the surface of a triangle in such wise that the area of the trapezium may be a mean proportional between the area of two triangles.

191. The perimeters of two similar triangles are proportional to the radii of the inscribed circles, and to the radii of the circumscribed circles. The areas of these triangles are proportional to the squares of the same radii.

192. Through a point situated on the bisector of an angle draw a secant, such that the part of that straight line comprised in the angle may be of a given length. This problem, in the particular case of the right angle, is known by the name of *Problem of Pappus*, a celebrated Greek geometer who lived in the fourth century.

193. Express the area of the circle in the function of its circumference. Apply the formula when found, calculating the area of a terrestrial meridian within a square kilomètre.

194. Given a regular hexagon A B C D E F: the summits are joined two and two by the diagonals A C, B D, C E, D F, E A, F B: first, prove that the polygon *a b c d e f* formed by the intersections of the consecutive diagonals is regular; second, find the ratio of the surface of this polygon to that of the given hexagon.

195. If, on one of the sides of the right angle of a right-angled triangle as diameter, semi-circumferences are described that shall be exterior to the triangle, each of these curves makes with the semi-circumference drawn through the summits of the triangle, a figure which has the shape of a crescent, and which is named the *Lunule* of *Hippocrates*, a Greek geometer of the fifth century. Prove that the sum of the surfaces of the two lunules is equivalent to the surface of the right-angled triangle.

196. Describe a circle which touches internally a given circle, and divides its surface into two parts proportional to two given lines.

197. The surface included between two concentric circumferences is equivalent to the circle, which has for diameter a chord of the external circumference, tangent to the internal circumference.

198. Describe two concentric circumferences, such that

the smaller divides into two equivalent parts the surface included in the greater.

199. What is the relation of the areas of the regular hexagons inscribed in and circumscribed round the same circle?

200. The area of a regular convex dodecagon is equal to three times the square of its radius.

201. The sum of the perpendiculars dropped from any point in the interior of a regular polygon on all the sides of this polygon is constant.

202. If the diameter A B of a circle be divided into any two segments A C, C B, and on one side of the straight line A B a semi-circumference be described on the first segment as diameter, and on the other side a semi-circumference on the second segment, the combined length of these two curves divides the surface of the given circle into two parts proportional to the segments A C, C B of the diameter A B.

203. Take on the circumference of a circle two arcs A B, B C, respectively equal to the fourth and the sixth of the circumference; then draw a secant through point A and the middle of the straight line B C; the chord intercepted on this secant represents, within a thousandth part of the radius, the side of the square equivalent to the circle.

204. If in a circle three radii O A, O B, O C are drawn, forming between them angles of 120°, and if in these straight lines lengths O A′, O B′, O C′ be taken, equal to the side of the square inscribed in the circle, the equilateral triangle A′ B′ C′ is equivalent to the regular hexagon inscribed in the same circle.

205. Construct seven equal regular hexagons, so that six of them may have two summits situated on a given circumference and a side common with the seventh, which must have the same centre as this circumference. Prove that the concave polygon formed by these seven hexagons is equivalent to the regular hexagon inscribed in the given circumference.

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